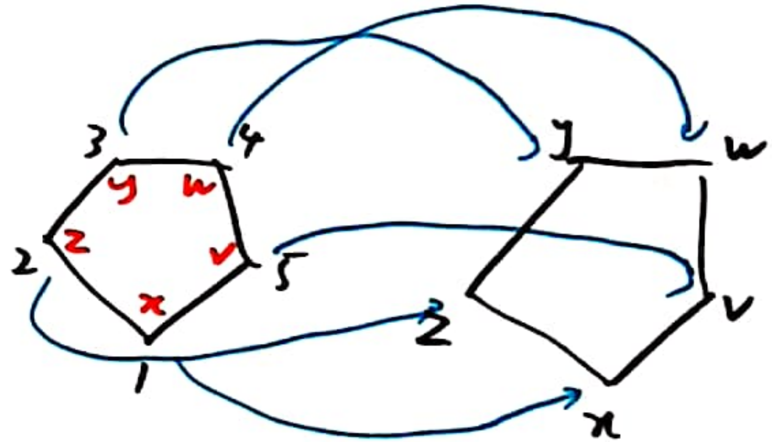
Expected number of copies of H in $G(n,p)$

<u>Recall</u> If G, H are graphs an isomorphism (IM) from G to H is a bijection $\phi : V(G) \to V(H)$ s.t. $\phi(i)\phi(j) \in E(H) \iff ij \in E(G)$

An automorphism of H is an IM $\phi : H \to H$.



We write $\mathrm{aut}(H)$ for # of automorphism.

$$\begin{tikzcd} G \arrow[r, "\phi"] \arrow[rr, bend right, "\psi \circ \phi"'] & H \arrow[r, "\psi"] & K \end{tikzcd}$$

Note if H_1, H_2 are isom. $\#$ IMs $H_1 \to H_3$

$= \#$ IMs $H_2 \to H_3$

$$\begin{tikzcd}
H_1 \arrow[d, bend right, "\psi"'] \arrow[dr, "\phi"] & \\
H_2 \arrow[r, "\phi \circ \psi^{-1}"'] & H_3
\end{tikzcd}$$

ϕ IM

$\Leftrightarrow \phi \circ \psi^{-1}$ IM

Also if $H_1, H_2 \cong H$ then $\exists$ ~~dat~~ exactly $\mathrm{aut}(H)$

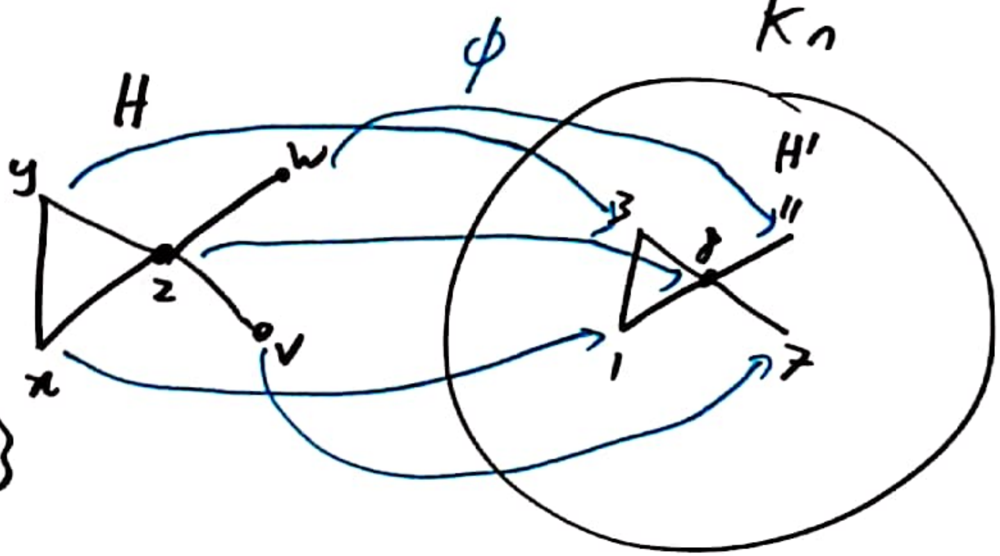
IMs $H_1 \to H_2$.

Each injection $\phi : V(H) \to [n]$

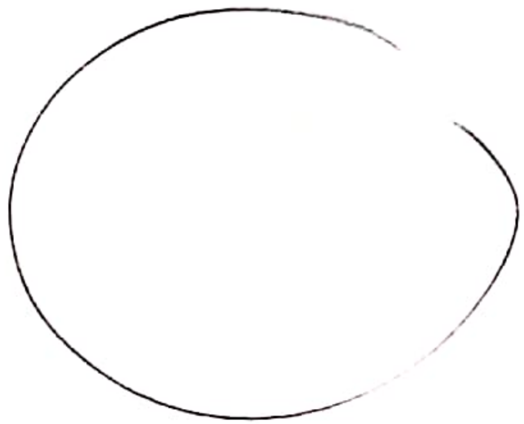
describes a copy of H,

with vertices $\mathrm{im}(\phi) = \{\phi(i) : i \in V(H)\}$

edges $\{\phi(i)\phi(j) : ij \in E(H)\}$



ϕ describes a particular copy H' $\Leftrightarrow$ ϕ is an IM $H \to H'$!



$\therefore$ by observation before $\exists$ exactly $\mathrm{aut}(H)$ injections ϕ describe H'

$\therefore$ # copies $= \dfrac{\# \text{inj}}{\text{aut}(H)} = \dfrac{n(n-1)(n-2)\cdots(n-V+1)}{\text{aut}(H)}$ $\quad V = |H|$

Now consider $X = X_H =$ # copies of H in $G(n,p)$

$$\mathbb{E}[X] = \frac{n(n-1)\cdots(n-v+1)}{\text{aut}(H)} p^e \qquad e = e(H)$$

(prob. a particular copy present is p^e)

Fix H, consider $p = p(n)$, $n \to \infty$

then $\mathbb{E}[X] \sim \dfrac{n^v p^e}{\text{aut}(H)} = \Theta(n^v p^e)$

Suggests threshold for copy of H to appear

is when $n^v p^e \approx 1$, i.e. $p = n^{-v/e}$.

For K_4 ✓

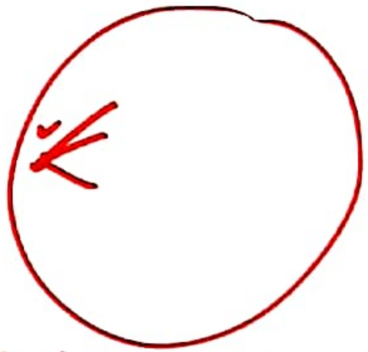
$v=4$

$e=6$

$n^{-4/6} = n^{-2/3}$

$\frac{e}{v} \approx 1$ $\quad \frac{v}{e} \approx 1$ say 0.99 threshold $n^{-0.99}$?

$\frac{e}{v} = \frac{7}{5} < \frac{6}{4}$ $\quad n^{-5/7} \ll n^{-2/3}$



$p = n^{-5/7}$

$Bi(n-1, p) \approx np = n^{2/7}$

Defn The edge density of H is $\dfrac{e(H)}{|H|} = d(H)$

Recall by handshaking $\sum_v d(v) = 2e(H)$

$$\frac{1}{|H|}\sum_v d(v) = 2d(H)$$

av. degree

H is balanced if $d(H') \le d(H)$ $\forall H' \subseteq H$

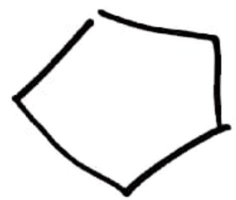
strictly balanced $<$ $\forall H' \subsetneq H$

eg strictly balanced :

K_n

$K_{a,b}$

cycles



trees

balanced

av degree 2

Theorem 2.5 Let H be a FIXED balanced graph with v vertices and e edges. Then $p^*(n) = n^{-v/e}$ is a threshold function for the property 'contains a copy of H' in $G(n,p)$.

Proof Let $p = p(n)$ Let $X = X_n$ = # copies of H in $G(n,p)$

Then $\mu = \mathbb{E}[X] = \Theta(n^v p^e)$.

(i) suppose $\frac{p}{p^*} \to 0$ then $pn^{v/e} \to 0$. $\therefore n^v p^e \to 0$

$\therefore \mathbb{E}[X] \to 0$ $\therefore \mathbb{P}(X \geq 1) \leq \mathbb{E}[X] \to 0$ ✓

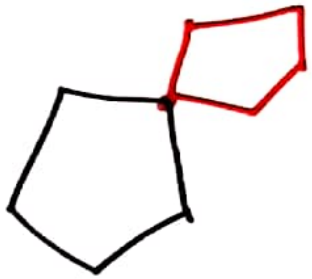
(ii) suppose $\frac{p}{p^*} \to \infty$. $\quad pn^{v/e} \to \infty \quad \therefore \; n^v p^e \to \infty$

$\therefore \; \mu \to \infty$

Plan use Cor 2.4
show $\Delta = o(\mu^2)$

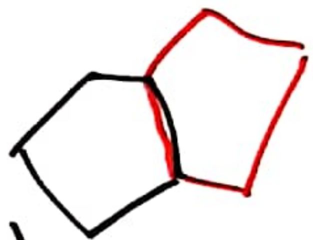
Let A_i be event that i^{th} possible copy H_i of H is present in $G(n,p)$

Write $i \sim j$ if H_i, H_j share ≥ 1 edge.



Let

$$\Delta = \sum_i \sum_{j \sim i} \mathbb{P}(A_i \cap A_j)$$



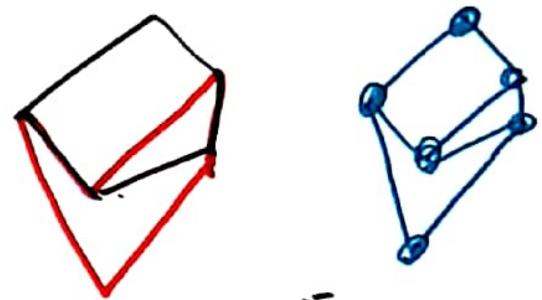
By Cor 2.4 it suffices to show $\Delta = o(\mu^2)$.

$A_i \cap A_j$ = event $H_i \cup H_j \subseteq G(n,p)$

$i \sim j$ if $H_i \cap H_j$ has ≥ 1 edge

$$\Delta = \sum_i \sum_{j \sim i} \mathbb{P}(H_i \cup H_j \subseteq G(n,p))$$

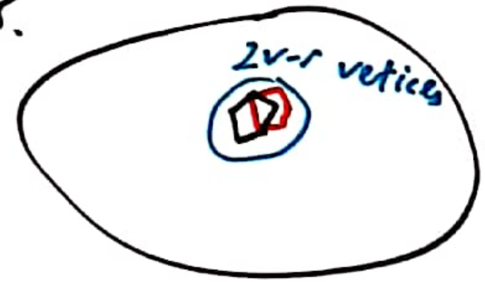
$$= \sum_i \sum_{j \sim i} p^{e(H_i \cup H_j)} \qquad (*)$$



$6^5 \times 6^5$

For $r \geq 2$, $s \geq 1$ consider all terms in $(*)$ where $H_i \cap H_j$ has r vertices, s edges

N.B.



$\Delta_{r,s}$ = sum of such terms

In each term $H_i \cup H_j$ has $2v-r$ vertices, $2e-s$ edges.

For given r,s there are $O(n^{2v-r})$ such terms

$$\therefore \Delta_{r,s} = O(n^{2v-r} p^{2e-s})$$

$$= O\left(\frac{\mu^2}{n^r p^s}\right)$$

$\mu = \Theta(n^v p^e)$

$n^r p^s \overset{?}{\to} \infty$

$n^v p^e \to \infty$

Now $H_i \cap H_j \subseteq H_i \subsetneq H$ which is balanced ok if $\frac{r}{s} \geq \frac{v}{e}$

$$\therefore d(H_i \cap H_j) \leq d(H)$$

i.e. $\frac{s}{r} \leq \frac{e}{v}$.

$$\therefore s \leq \frac{re}{v}$$

+ve constant

$$\therefore n^r p^s \geq n^r p^{\frac{re}{v}} = \underbrace{(n^v p^e)}_{\to\infty}{}^{\frac{r}{v}} \to \infty$$

$\therefore \quad \Delta_{r,s} = o(\mu^2)$

$v <$ $e <$ CONSTANT

$$\therefore \quad \Delta = \sum_{r=2}^{v} \sum_{s=1}^{e} \Delta_{r,s} = o(\mu^2)$$

by Cor 2.4 $\quad \mathbb{P}(X \geq 1) \to 1.$ $\square$

Remark: in general (any fixed H)

threshold is $n^{-d(H')}$ where H' is a densest subgraph of H

$$\frac{p}{p^*} \to \infty \qquad \frac{p}{p^*} \to 0$$

$$\frac{p}{p^*} \to c \qquad \mu \to d \qquad \mathbb{P}(X \geq 1) \to 1 - e^{-d}.$$

Dependence among events.

Standard situation: $A_1, \dots, A_n$ are 'bad' events. We want conditions that tell us $\mathbb{P}(\bigcap_{i=1}^{n} A_i^c) > 0$

(i) If $\sum \mathbb{P}(A_i) < 1$ then done by Union bound whatever the relationship between the A_i

(ii) If $A_1, \dots, A_n$ independent

$$\mathbb{P}(\cap A_i^c) = \prod \mathbb{P}(A_i^c) = \prod \underbrace{(1 - \mathbb{P}(A_i))}_{>0} > 0$$

<u>Defn</u> $\mathbb{P}(A|B) = \dfrac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$ Undefined if $\mathbb{P}(B) = 0$

<u>OR</u> defined by $\mathbb{P}(A \cap B) = \mathbb{P}(A|B)\,\mathbb{P}(B)$ - ALWAYS TRUE if $\mathbb{P}(B) = 0$

Can think of $\mathbb{P}(A|B) \begin{smallmatrix} \geq x \\ = x \\ \leq x \end{smallmatrix}$ as short for

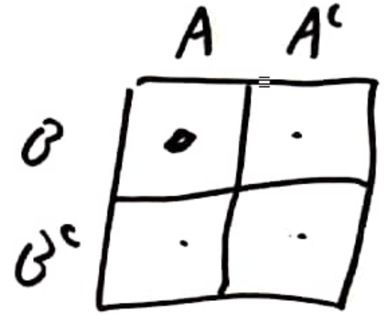
$$\mathbb{P}(A \cap B) \begin{smallmatrix} \geq \\ = \\ \leq \end{smallmatrix} x\,\mathbb{P}(B)$$

true (by convention) if $\mathbb{P}(B) = 0$.

Recall two events A, B are independent if

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\,\mathbb{P}(B) \qquad \left[\mathbb{P}(A \mid B) = \mathbb{P}(A)\right]$$

$$\Rightarrow \mathbb{P}(A \cap B^c) = \mathbb{P}(A)\,\mathbb{P}(B^c)$$



Defn A family $(A_1, \dots, A_n)$ of events is independent if $\forall$ disjoint $S, T \subseteq [n]$

$$\mathbb{P}\Big(\bigcap_{i\in S} A_i \cap \bigcap_{i \in T} A_i^c\Big) = \prod_{i\in S} \mathbb{P}(A_i) \prod_{i \in T} \mathbb{P}(A_i^c)$$

$$\Big[\prod_{i\in\emptyset} \sim = 1; \quad \bigcap_{i\in S_{\emptyset}} A_i = \{\omega \in \Omega : \forall i \in \overset{\emptyset}{S} \ \omega \in A_i\} = \Omega$$

$$\mathbb{P}(A_1 \cap A_3^c \mid A_2 \cap A_5^c \cap A_7) = \mathbb{P}(A_1 \cap A_3^c)$$

<u>Defn</u> An event A is indep. of a family $(B_1, \dots, B_n)$ if

$\forall$ disjoint $S, T \subseteq [n]$

$$\mathbb{P}\Big(A \,\Big|\, \bigcap_{i \in S} B_i \cap \bigcap_{i \in T} B_i^c\Big) = \mathbb{P}(A)$$

~~Telling~~ Knowing whether or not some B_i hold has no effect on prob. of A.

$\boxed{\text{FACT}}$ independence $\overset{\not\Leftarrow}{\Rightarrow}$ pairwise indep.

$$A \text{ indep. of } \{B, C\} \underset{\Leftarrow}{\Rightarrow} \begin{cases} A, B \text{ indep.} \\ A, C \text{ indep.} \end{cases}$$

eg consider X, Y, Z iid with $\mathbb{P}(X=1) = 1/2 = \mathbb{P}(X=0)$

Let $A = \{X = Y\}$, $B = \{Y = Z\}$ $C = \{Z = X\}$

Easy to see pairwise indep.

BUT $\mathbb{P}(A \mid B \cap C) = 1$

A digraph is like a graph but 

with arrows: an edge has a direction $i \to j$

Allow 

Don't allow  

A digraph D on $[n]$ is a dependency digraph for events $A_1, \dots, A_n$ if for each i A_i is indep. of

$$\{A_j : j \neq i,\ i \not\to j\}$$

A_1 is indep. of $\{A_2, A_5, A_6\}$

A_1 i 'allowed' to depend on A_j

$A_3, A_4\}$ if $i \to j$ }

Theorem 3.1 (Local Lemma, General Form).

Let D be a dependency digraph for events $A_1, \dots, A_n$.

Suppose $\exists\ 0 \leq x_i \leq 1$ s.t.

$$\forall i \quad \mathbb{P}(A_i) \leq x_i \prod_{j:\, i \to j} (1 - x_j).$$

Then $\mathbb{P}\left(\bigcap_{i=1}^{n} A_i^c\right) \geq \prod_{i=1}^{n} (1 - x_i) > 0$

Proof $\mathbb{P}(\bigcap_{i=1}^{n} A_i^c) = \mathbb{P}(A_1^c)\,\mathbb{P}(A_2^c \mid A_1^c)\,\mathbb{P}(A_3^c \mid A_1^c \cap A_2^c) \cdots$

We CLAIM : for any $S \subseteq [n]$ and any $i \notin S$ we have

$$\mathbb{P}(A_i^c \mid \bigcap_{j \in S} A_j^c) \geq 1 - x_i \qquad (1)$$

$$\mathbb{P}(A_i \mid \bigcap_{j \in S} A_j^c) \leq x_i \qquad (2)$$

If so, done ✓

Proof of claim: induction on $|S|$.

$\bigcap_{j \in S} A_j^c = \Omega$

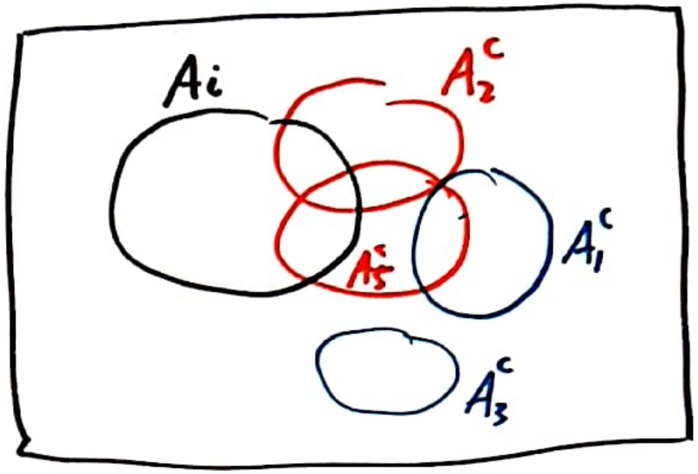
Base case $|S| = 0$

$$\mathbb{P}(A_i \mid \cdots) = \mathbb{P}(A_i) \leq x_i \prod_{i \leftrightarrow j} (1 - x_j) \leq x_i \quad ✓$$

Suppose $|S| = r$, $i \notin S$ & (1), (2) hold $\forall S', i'$ with $|S'| < r$.

Let $S_0 = \{ j \in S : i \not\to j \}$

$S_1 = \{ j \in S : i \to j \}$



Let $B = \bigcap_{j \in S_1} A_j^c$ $\quad C = \bigcap_{j \in S_0} A_j^c$

Wnt $\mathbb{P}(A_i \mid B \cap C) \le x_i$

Since $S_0 \subseteq \{ j : j \neq i \;\; i \not\to j \}$ by defn of dependency digraph, A_i indp of C $\therefore$ $\mathbb{P}(A_i \mid C) = \mathbb{P}(A_i)$

$$\mathbb{P}(A_i \mid B\cap C) = \frac{\mathbb{P}(A_i\cap B\cap C)}{\mathbb{P}(B\cap C)} = \frac{\mathbb{P}(A_i\cap B\cap C)}{\mathbb{P}(C)}\,\frac{\mathbb{P}(C)}{\mathbb{P}(B\cap C)} = \frac{\mathbb{P}(A_i\cap B\mid C)\ *}{\mathbb{P}(B\mid C)\ (\dagger)}$$

$$*\ \mathbb{P}(A_i\cap B\mid C) \le \mathbb{P}(A_i\mid C) = \mathbb{P}(A_i) \le x_i \prod_{j:i\to j}(1-x_j) \quad (3)$$

$(\dagger)$ Say $S_0 = \{1,3\}$ $S_1 = \{2,5\}$

$$\mathbb{P}(A_2^c\cap A_5^c \mid A_1^c\cap A_3^c) = \mathbb{P}(A_2^c \mid A_1^c\cap A_3^c)\,\mathbb{P}(A_5^c \mid \underbrace{A_1^c\cap A_3^c\cap A_2^c}_{\le r-1})$$

By IH (1) $\ge (1-x_2)(1-x_5)$

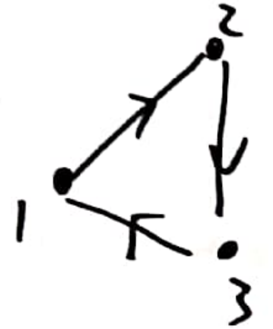
$$\mathbb{P}(B\mid C) \ge \prod_{j\in S_1}(1-x_j) \ge \prod_{j:i\to j}(1-x_j) \quad (4)$$

since $S_1 \subseteq \{j : i\to j\}$. Combine $\mathbb{P}(\ \ ) \ge \frac{3}{4} = x_i$ $\square$

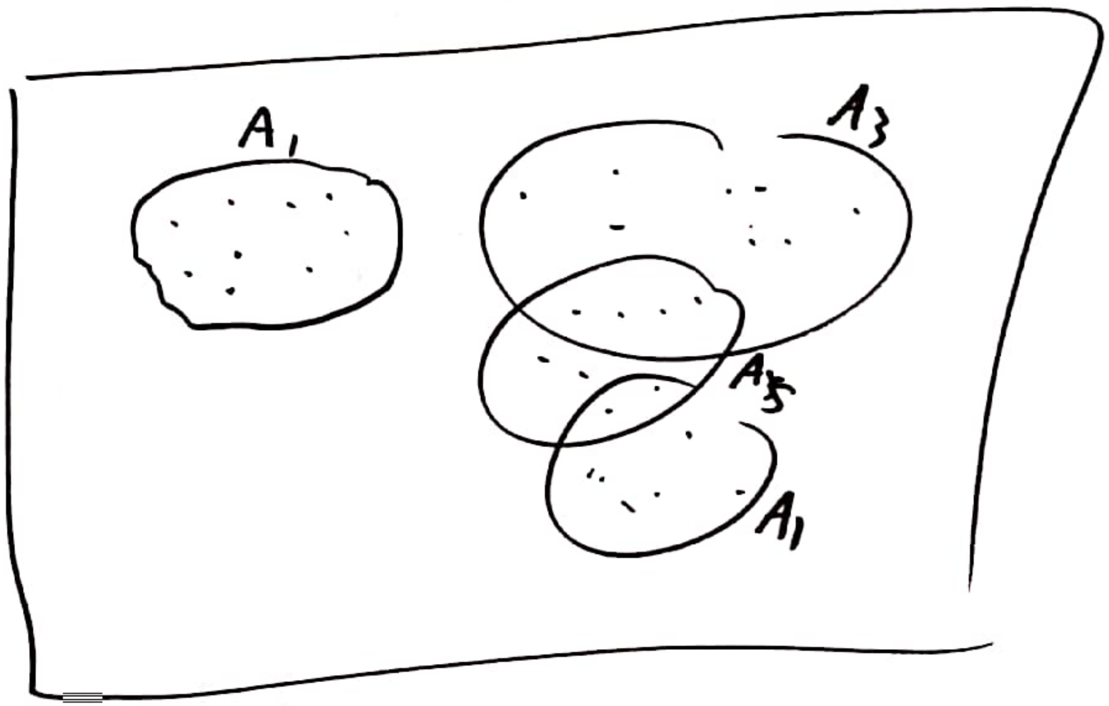
Consider example where A_1, A_2, A_3 are pairwise indep

but not indep.

A_i indep of $\{A_j : j \neq i, i \not\to j\}$



A_1 indep of $\{\cancel{A_2}, A_3\}$



$F = E(K_n)$ $X_e = \begin{cases} 1 & e \text{ present} \\ 0 & \text{absent} \end{cases}$ in $G(n,p)$

<u>Lemma 3.2</u> Let $(X_\alpha)_{\alpha \in F}$ be a family of indep. RVs.

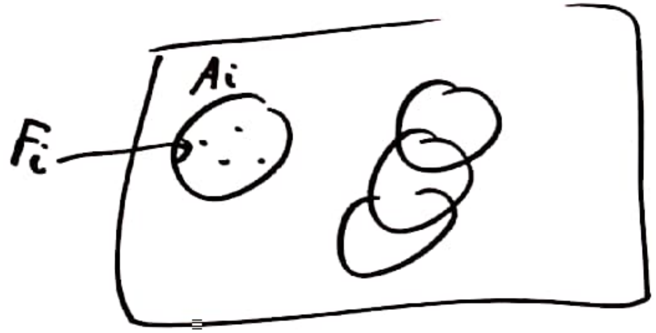
Suppose $A_1, \dots, A_n$ are events where A_i is determined by $(X_\alpha : \alpha \in F_i)$ for $F_i \in F$.

Form a digraph D by joining $i \to j$, $j \to i$ if $F_i \cap F_j \neq \emptyset$. Then D is a dependency digraph for $A_1 \dots A_n$.

<u>Proof</u> Consider $i \in [n]$. Need A_i is indep of

$$\mathcal{B}_i = \{A_j : j \neq i \quad i \not\to j\} = \{A_j : F_i \cap F_j = \emptyset\}$$

The events in $\mathcal{B}_i$ are determined by the X_α $\alpha \in \bigcup_{j : F_i \cap F_j = \emptyset} F_j$ which is disjoint from F_i. $\therefore$ A_i is indep of $\mathcal{B}_i$ $\square$



Thm 3.3 Local lemma - symmetric version

Let D be a dependency digraph for events $A_1, \dots, A_n$ in which all out-degrees are $\leq d$. Suppose

$$\mathbb{P}(A_i) \leq p \quad \forall i \quad \text{where} \quad ep(d+1) \leq 1 \quad \text{Then} \quad \mathbb{P}(\cap A_i^c) > 0.$$

$\#j : i \to j$

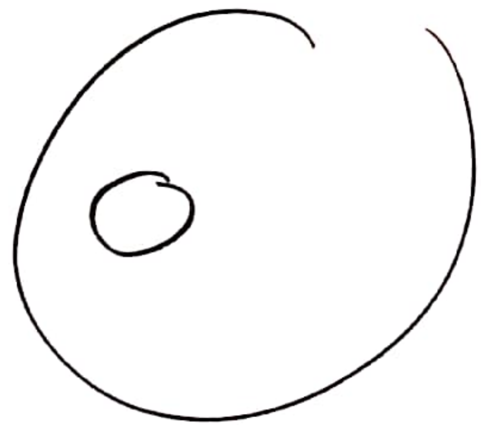
Pf. Choose $x_i = \frac{1}{d+1}$.

$$x_i \prod_{i \to j} (1 - x_j) \geq \cancel{x(1-x)^d} = \frac{1}{d+1} \left(\frac{d}{d+1} \right)^{\#j : i \to j}$$

$$\geq \frac{1}{d+1} \left(\frac{d}{d+1} \right)^d \geq \frac{1}{e(d+1)} \geq p \quad \therefore \text{Thm 3.1 applies} \quad \square$$

$$\left(\frac{d+1}{d} \right)^d = \left(1 + \frac{1}{d} \right)^d \leq e^{\frac{1}{d} \cdot d} = e \qquad 1 + z \leq e^z$$

$$\forall i \quad \mathbb{P}(A_i) \le x_i \prod_{i \to j} (1 - x_j)$$



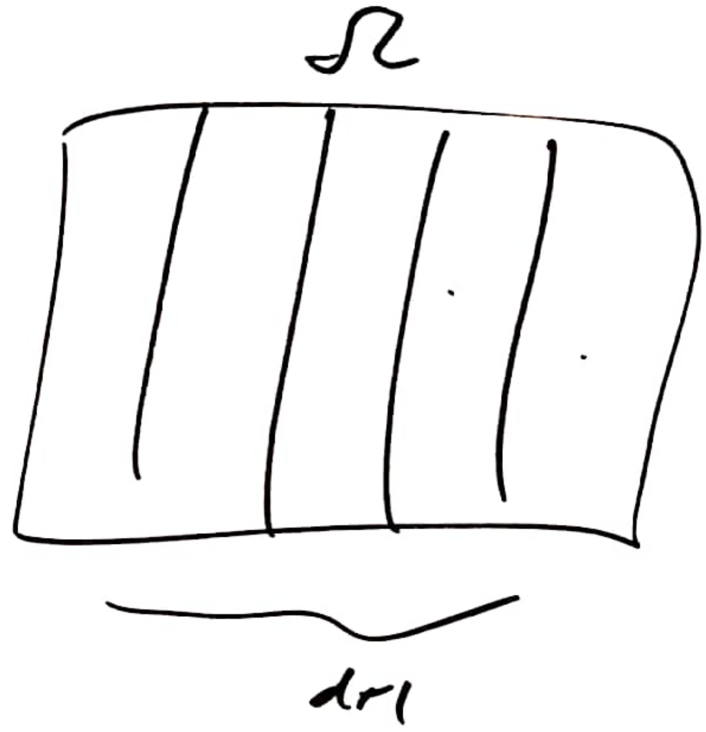
$$p \le x \prod_{i \to j} (1 - x)$$

$$p \le x(1-x)^d \qquad\qquad x = \frac{1}{d+1}$$

$$p \le \frac{1}{d+1} \left(\frac{d}{d+1} \right)^d$$

Needed $ep(d+1) \leq 1$ $\qquad p \leq \frac{1}{e(d+1)}$

$\mathbb{P}(A_i) = \frac{1}{d+1}$



$A \cap B_j | C$ $\qquad A | C$