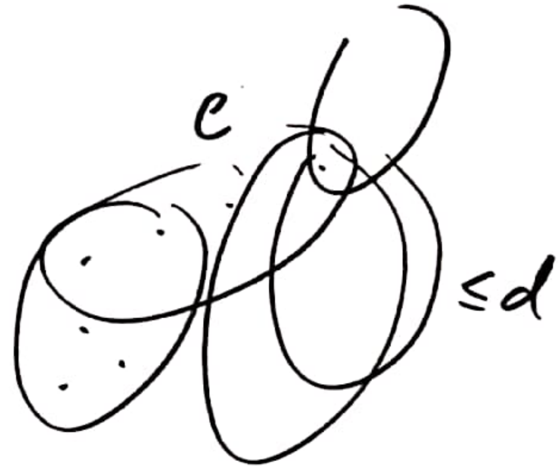
Theorem 3.4 Let H be an r-uniform hypergraph in which each edge meets $\leq d$ other edges.

If $d+1 \leq \frac{2^{r-1}}{e}$ then H is 2-colourable



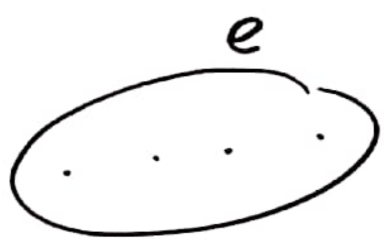
Proof Colour vertices of H as before, independently, each red/blue with $\frac{1}{2}$.

Let $A_e = \{$edge e is monochromatic$\}$

$$\mathbb{P}(A_e) = \frac{1}{2^{r-1}}$$

let $(X_v)_{v\in V(H)}$ be the indep RVs giving colours of the vertices. For each edge e let

$F_e = e = \{\text{vertices in } e\}$. Then



whether A_e holds determined by $(X_v)_{v\in F_e}$. $(X_v)_{v\in e}$

Form a digraph D on $E(H)$ by joining $e \rightleftarrows f$ if $F_e \cap F_f \neq \emptyset$ ie. if e,f intersect.

By Lemma 3.2 D is a dep. digraph for (A_e)

By assumption all degrees in $D \le d$. $\mathbb{P}(A_e) = p$ $\forall e$ $p = \frac{1}{2^{r-1}}$

& $ep(d+1) \le \frac{e}{2^{r-1}} \frac{2^{r-1}}{e} = 1$ $\therefore$ by Lemma 3.3 $\mathbb{P}(\cap A_e^c) > 0$ $\square$

'edge degree' $\leq \frac{2^{r-1}}{e}$ $\Rightarrow$ 2-col.

total # edges $\leq 2^{r-1}$ $\Rightarrow$ 2-col.

$m \leq 10$

$\Delta \leq 4$

__Theorem 3.5__ Suppose $n, k \geq 3$ satisfy $e \cdot 2^{1-\binom{k}{2}} \binom{k}{2} \binom{n}{k-2} \leq 1$

Then $R(k,k) > n$

__Proof__ Colour edges of K_n indep., each red/blue with $\frac{1}{2}$.

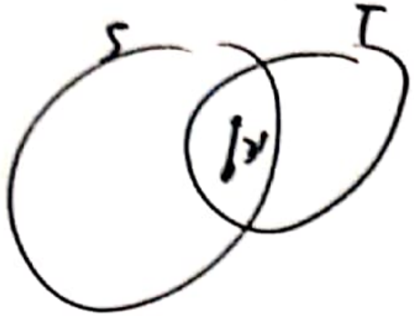
For each $S \subseteq V(K_n)$ of size k let $A_S = \{S$ spans a mono. $K_k\}$. $\mathbb{P}(A_S) = 2^{1-\binom{k}{2}} = p$.

Here underlying indep RVs are $(X_e)_{e \in E(K_n)}$ giving colours of edges. For each S in Lemma 3.2 language $F_S = \{$all $\binom{k}{2}$ edges in $S\}$

S

Form a ~~dep~~ digraph D by joining $S \rightleftarrows T$ if $F_S \cap F_T \neq \emptyset$

i.e. if

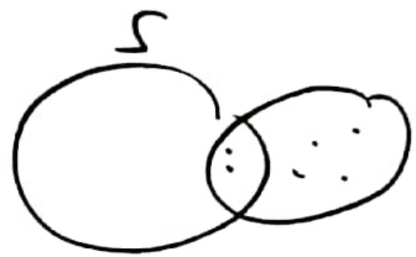


i.e. $|S \cap T| \geq 2$

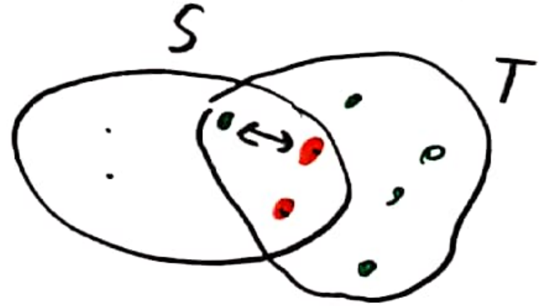
By Lemma 3.2 D is a valid dep. digraph.

How many T meet a given S in ≥ 2 vxs?

$$\cancel{\binom{k}{2}\binom{n-k}{k-2} + \binom{k}{3}\binom{n-k}{k-3} + \cdots}$$



$$\leq \binom{k}{2}\binom{n}{k-2} - 1 = d$$



$ep(d+1) \leq 1$ by assumption

By Theorem 3.3 $\mathbb{P}(\bigcap A_S^c) > 0$ i.e. $\exists$ a good colouring.

Corollary 3.6

$$R(k,k) \geq \sqrt{2}\,\frac{k}{e}\,2^{k/2}\,(1+o(1)).$$

Pf Calculation. $\square$

Union bound	$\frac{1}{\sqrt{2}}\left(\qquad \right)$
Deletion bound	$1 \cdot \left(\qquad \right)$

$\leq 4^k$

$R(3,3) = 6$

$R(4,4) = 18$

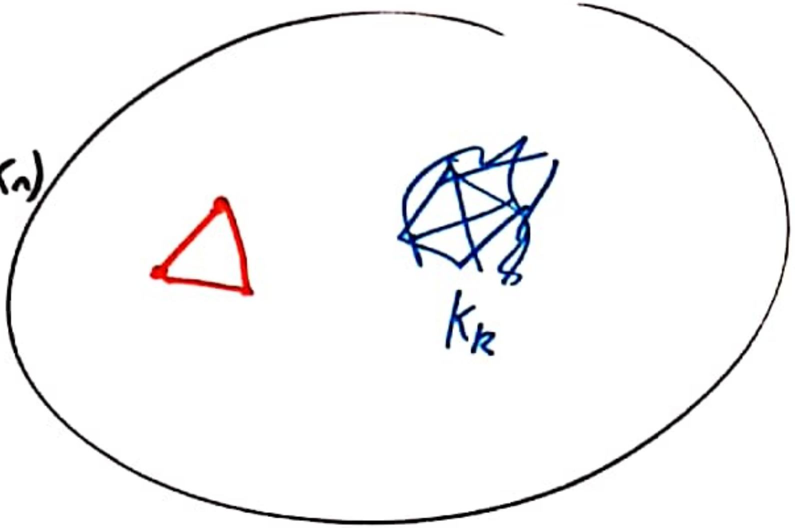
$R(5,5)$ 43-48

$R(6,6)$ 102 - 165

Theorem 3.7 $\exists\, c>0$ s.t. $R(3,k) \geq \dfrac{ck^2}{(\log k)^2}$ if k is large enough

Idea consider suitable $k = k(n)$

Aim show $\exists$ 'good' colouring of $E(K_n)$



Colour randomly: each edge red with some probability $p = p(n) \to 0$, blue otherwise, independently.

For each S with $|S| = 3$ let $A_S = \{S \text{ spans a red triangle}\}$

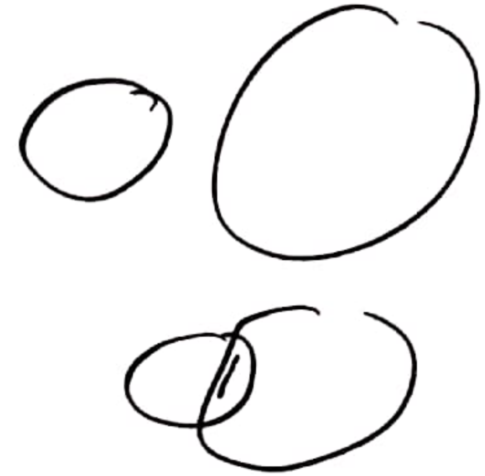
T $\quad |T| = k \quad$ $B_T = \{T \text{ spans a blue } K_k\}$

Aim $\mathbb{P}\left(\bigcap_S A_S^c \cap \bigcap_T B_T^c\right) > 0$

As before can build a dp. digraph D by joining

$S \sim T$ if share ≥ 2 vxs

$S \sim S'$ " "



$\mathbb{P}(A_i) \leq x_i \prod_{j: i \to j} (1-x_j)$ ...

Choose $x_i = x$ for all 'A events'

$x_i = y$ " 'B events'

$\mathbb{P}(A_S) = p^3$ $\quad \mathbb{P}(B_T) = (1-p)^{\binom{k}{2}}$

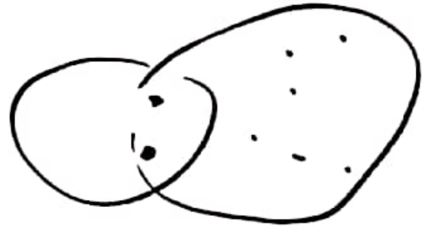
Need eg $\forall S$ $\quad p^3 \leq x(1-x)^{\# S' : S \to S'} (1-y)^{\# T : S \to T}$

OK if $p^3 \leq x(1-x)^{\text{overestimate}} (1-y)^{\text{overestimate}}$

Each S joined to $3(n-3) \le 3n$ S'

Each S joined to $\le \binom{n}{k} \le n^k$ T

Since there are only this many sets T

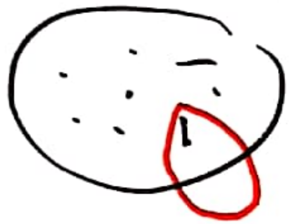


Each T joined to $\le \binom{k}{2} n$ S

$\le n^k$ T'

$3\binom{n-3}{k-2}$

+



Done if

$$3p^3 \le x(1-x)^{3n}\,\cancel{(1-y)^{n^k}}$$

$$\& \quad 3(1-p)^{\binom{k}{2}} \le y\,(1-\underset{x}{\cancel{y}})^{\binom{k}{2}n}\,\cancel{(1-y)^{n^k}}$$

If so $\exists$ good colouring $\therefore$ $R(3,k) > n$

Take $y = n^{-k}$. Then $(1-y)^{n^k} \to \frac{1}{e} > \frac{1}{3}$

$(1-x)^{3n}$

Done if $\cancel{9\cancel{8}p^3 \leq x(1-x)^{3n}}$ $\qquad$ $9p^3 \leq x$

$$3(1-p)^{\binom{k}{2}} \leq n^{-k}(1-x)^{\binom{k}{2}n} \quad (*)$$

$p \to 0$

Suppose $\cancel{x \geq \frac{1}{3n}}$, $\cancel{x \to 0}$

$-\log(1-p) = p + O(p^2)$

$\sim p$

$-\log$

$$C + \binom{k}{2}p \geq k\log n + \binom{k}{2}nx$$

$$p \geq \frac{2\log n}{k-1} + nx$$

$a \geq b + c$

$\Uparrow$

$a \geq 2b$ & $a \geq 2c$

$p \geq \frac{5\log n}{k}$, $p \geq 3nx$

$x \leq \frac{p}{3n} \leq \frac{1}{3n}$

$x \geq 9p^3$, $\quad p \geq 3n\,x \quad$ $p \geq \frac{5\log n}{k}$

$$\frac{p}{3n} \geq x \geq 9p^3$$

can solve $\iff$ $\frac{p}{3n} \geq 9p^3$ i.e. $\frac{1}{27n} \geq p^2$

Take $p = \frac{1}{6\sqrt{n}}$. Need $k \geq \frac{5\log n}{p} = 30\sqrt{n}\log n$

so take $k = \lceil 30\sqrt{n}\log n \rceil$

then $R(3,k) > n$.

Show for some $k \sim 30\sqrt{n}\log n$ $\quad R(3,k) > n$

$$\Downarrow$$

$$\log k = O(1) + \tfrac{1}{2}\log n + \log\log n$$

$$\sim \tfrac{1}{2}\log n$$

$$k \sim 30\sqrt{n}\cdot 2\log k$$

$$\sqrt{n} \sim \frac{k}{60\log k} \qquad\qquad \frac{k^2}{\log k}$$

$$n \sim \frac{k^2}{C(\log k)^2} \qquad \square$$

Standard situation: sequence E_n of events

want $\mathbb{P}(E_n) \to 1$

$\to 0$

eg. if $\mathbb{P}(E_n) \to 0$ $\mathbb{P}(F_n \to 0)$ $\mathbb{P}(G_n \to 0)$ $\Rightarrow$ $\mathbb{P}(E_n \cup F_n \cup G_n) \to 0$

What if # events grows with n?

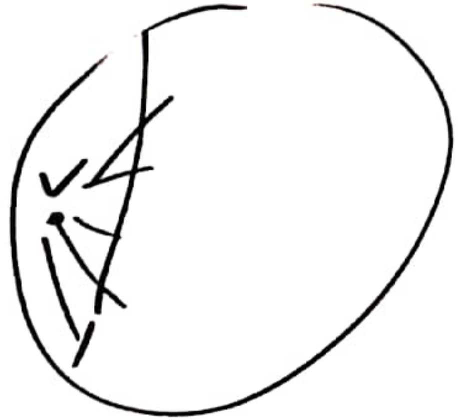
Consider $G(n,p)$, let $\Delta = \Delta(G(n,p))$ $p = \frac{1}{2}$.

$$\Delta(G) \geq d \iff \exists_v \text{ s.t. } d_v \geq d$$

$$\mathbb{P}(\qquad) \leq \sum_v \mathbb{P}(d_v \geq d) = n \underbrace{\mathbb{P}(d_v \geq d)}_{\text{want. } = o(\frac{1}{n})}$$

$\to 0$

For fixed v $\quad d_v \sim Bi(n-1,p) \le Bi(n,p)$



If $X \sim Bi(n,p)$ $\quad \mathbb{P}(d_v \ge d) \le \mathbb{P}(X \ge d)$

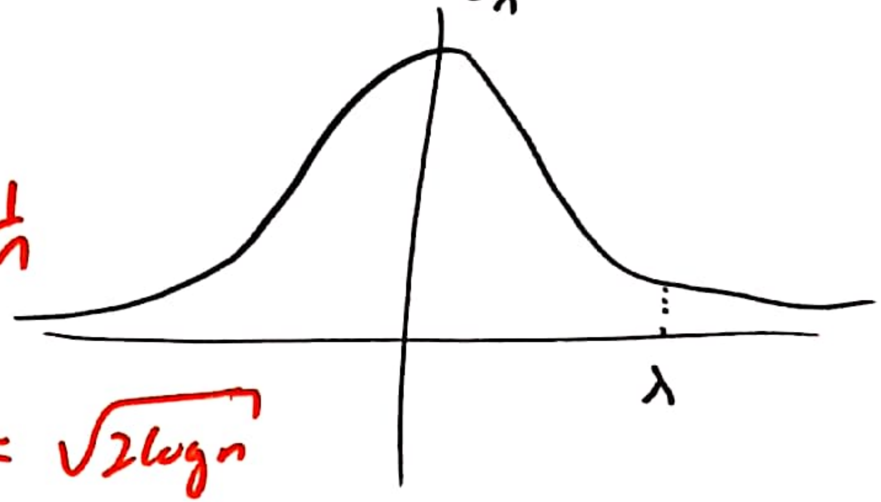
Say $p = \frac{1}{2}$. Then $\mu = \mathbb{E}X = \frac{n}{2}$ $\qquad \sigma^2 = \mathrm{Var}\, X = np(1-p) = \frac{n}{4}$

Chebyshev says $\qquad$ i.e. $\sigma = \frac{\sqrt{n}}{2}$

$$\mathbb{P}(X \ge \underbrace{\mu + \lambda\sigma}_{\gg n}) \le \frac{\mathrm{Var}\, X}{\lambda^2 \sigma^2} = \frac{1}{\lambda^2} \propto \frac{1}{n} \quad \text{need } \lambda \gg \sqrt{n}$$

$$\mathbb{P}\left(\frac{X-\mu}{\sigma} \geq \lambda\right) \leq \frac{1}{\lambda^2}$$

$$\frac{X_n-\mu_n}{\sigma_n} \to N(0,1)$$



$$\mathbb{P}\left(N(0,1) \geq \lambda\right) \lesssim e^{-\lambda^2/2} \approx \frac{1}{n}$$

if $\lambda \approx \sqrt{2\log n}$

CLT SUGGESTS STATES

$$\mathbb{P}\left(\frac{X_n-\mu_n}{\sigma_n} \geq \lambda\right) \overset{\longrightarrow}{\approx} \mathbb{P}\left(N(0,1) \geq \lambda\right) \approx e^{-\lambda^2/2}$$

if λ fixed as $n \to \infty$

Theorem 4.1 Suppose $n \geq 1$, $p, x \in (0,1)$. Let $X \sim \text{Bin}(n,p)$.

Then

$$\mathbb{P}(X \geq nx) \leq \left[\left(\frac{p}{x}\right)^x \left(\frac{1-p}{1-x}\right)^{1-x}\right]^n \quad \text{if } x \geq p$$

$$\mathbb{P}(X \leq nx) \leq \quad \text{"} \quad \text{if } x \leq p$$

Proof Idea: apply Markov to e^{tX} for suitable t.

(Need to know $\mathbb{E}[e^{tX}]$

We can write X as $X_1 + + \cdots + X_n$ where each X_i is $\begin{matrix} 1 & \text{w prob. } p \\ 0 & 1-p \end{matrix}$

& the X_i are independent.

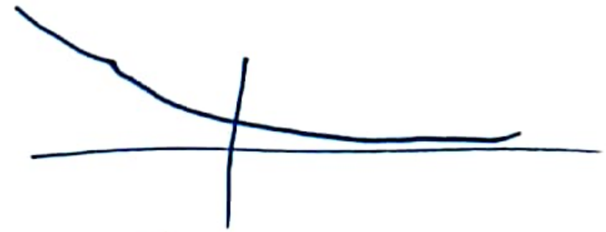
$$\mathbb{E}[e^{tX}] = \mathbb{E}[e^{tX_1 + \dots + tX_n}]$$

$$= \mathbb{E}[e^{tX_1} \cdot \dots \cdot e^{tX_n}]$$

$$= \mathbb{E}[e^{tX_1}] \cdots \mathbb{E}[e^{tX_n}] \quad \text{by indpendence.}$$

$$= (pe^t + 1 - p)^n$$

$$e^{tX_1} = \begin{cases} e^t & \text{with prob. } p \\ e^0 & \text{"} \quad 1-p \end{cases}$$



For $t \overset{<}{>} 0$ then $y \mapsto e^{ty}$ is strictly $\overset{\text{decreasing}}{\text{increasing}}$

so $X \overset{\le}{\ge} nx \iff \overbrace{e^{tX}}^{>0} \ge e^{tnx}$

By Markov $\mathbb{P}(\quad) \le \dfrac{\mathbb{E}[e^{tX}]}{e^{tnx}} = \left[(pe^t + 1 - p)e^{-tx}\right]^n.$

min $\quad (pe^{t} + (1-p))e^{-tx} = pe^{t(1-x)} + (1-p)e^{-tx}$

$\frac{d}{dt}$: $\quad p(1-x)e^{t(1-x)} = x(1-p)e^{-tx}$

$$e^{t} = \frac{e^{t(1-x)}}{e^{-tx}} = \boxed{\frac{x}{p}\,\frac{(1-p)}{1-x}} \qquad t = \log \boxed{} > 0$$

> 1

Suppose $x > p$: $\quad \frac{x}{p} > 1 \quad \frac{1-x}{1-p} < 1$

Then $\quad t = \log\left(\frac{x}{p}\,\frac{1-p}{1-x}\right) > 0$ is a valid choice.

$$\mathbb{P}(X \geq nx) \leq \left[(pe^t + 1-p) e^{-tx} \right]^n \qquad e^t = \frac{x}{p} \frac{1-p}{1-x}$$

$$\downarrow$$

$$\underbrace{p\left(\frac{x}{p} \frac{1-p}{1-x} + 1-p\right)}_{} \left(\frac{p}{x}\right)^x \left(\frac{1-x}{1-p}\right)^x$$

$$(1-p)\left[\frac{x + 1-x}{1-x}\right]$$

$$\left(\frac{1-p}{1-x}\right)^{1-x} \left(\frac{p}{x}\right)^x \qquad \checkmark \qquad \text{for } x > p.$$

$x > p$

If $X \sim Bin(n,p)$ = # success

Let $Y = n - X \sim Bin(n, 1-p)$ = # failures

For $x < p$

$$\mathbb{P}(X \leq nx) = \mathbb{P}(Y \geq n(1-x))$$

$$1-x \geq 1-p$$

so can apply previous case.

<u>Cor 4.2</u> Let $X \sim \mathrm{Bin}(n,p)$. Then for $h, t \geq 0$

$$X \geq n(p+h)$$

$$\mathbb{P}(X \geq np + nh) \leq e^{-2h^2 n} \qquad t = nh \qquad h = n/t$$

$$\mathbb{P}(X \geq np + t) \leq e^{-2t^2/n}$$

Also, for $0 \leq \varepsilon \leq 1$

$$\mathbb{P}(X \geq (1+\varepsilon)\underbrace{np}_{\mu}) \leq e^{-\varepsilon^2 np/4}$$

$$\mathbb{P}(X \leq (1-\varepsilon)np) \leq e^{-\varepsilon^2 np/2}$$

Suppose $p \to 0$ $\quad \mathrm{Var}(X) = \sigma^2 = np(1-p) \sim np$

so $\sigma \sim \sqrt{np}$. $\quad \varepsilon np = \lambda\sigma$ if $\lambda \sim \varepsilon\sqrt{np}$

Get $\mathbb{P}(X \geq \mu + \lambda\sigma) \leq e^{-\lambda^2/4}$ $\qquad \varepsilon \sim \frac{\lambda}{\sqrt{np}}$

$\quad X \leq \mu - \lambda\sigma) \leq \boxed{e^{-\lambda^2/2}}$

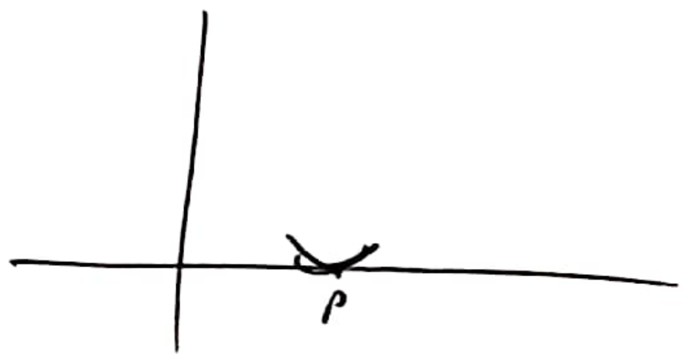
Pf: Fix $0<p<1$. For $x>p$ $x<p$ Theorem 4.1 says

$$\begin{matrix}\mathbb{P}(X \geq nx) \\ \mathbb{P}(X \leq nx)\end{matrix} \leq \exp(-f(x)n) \quad \text{where } f(x) = -\log[\quad]$$

$$f(x) = x\log\left(\frac{x}{p}\right) + (1-x)\log\left(\frac{1-x}{1-p}\right).$$

Note $f(p) = 0$

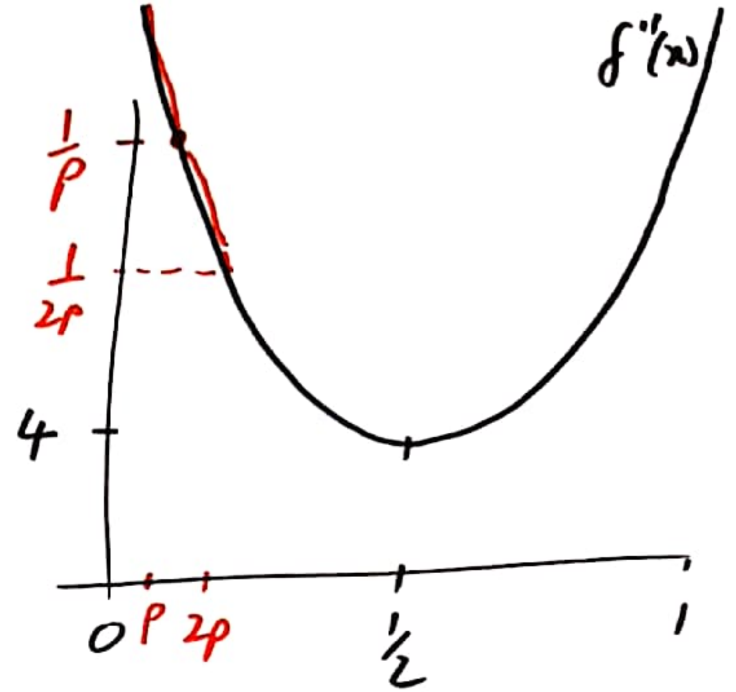
$$f'(x) = \log\left(\frac{x}{p}\right) - \log\left(\frac{1-x}{1-p}\right)$$



Note $f'(p) = 0$

$$f''(x) = \frac{1}{x} + \frac{1}{1-x}$$

$f(p) = f'(p) = 0 \qquad f''(x) = \frac{1}{x} + \frac{1}{1-x}$



Note $\forall x \; f''(x) \geq f''(\tfrac{1}{2}) = 4$.

By Taylor's Theorem, if $f''(x) \geq a$ on $[p, p+h]$ then $f(p+h) \geq \frac{a}{2}h^2$.

For first statement $f(p+h) \geq 2h^2$ ✓

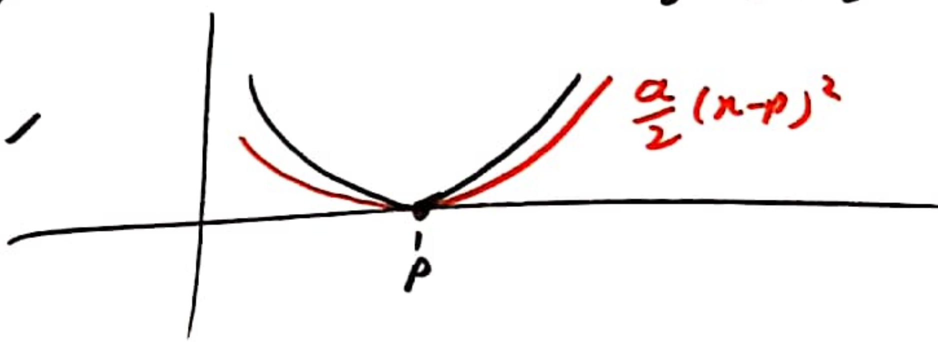
$\mathbb{P}(X \geq np + nh) \leq e^{-f(p+h)n} = e^{-2h^2 n}$ ✓

$p, p+h \qquad f''(x) \geq \frac{1}{x} \geq \frac{1}{p}$

For third on $[p, p+h)$ $f''(x) \geq \frac{1}{x} \geq \frac{1}{p+h} \geq \frac{1}{2p}$

Here $h = \varepsilon p \qquad \cancel{f''(x) \geq}\; p+h \leq 2p$

$f(p+h) \leq \frac{1}{4p}\varepsilon^2 p^2 = \frac{\varepsilon^2 p}{4}$



$\mathbb{P}(X \geq np + \varepsilon np$

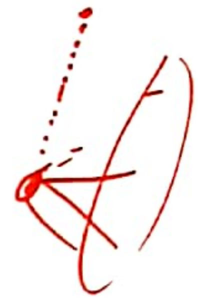
Thm 4.3 Let $p = p(n)$ satisfy $np \geq 10 \log n$. Let $\Delta = \Delta(G(n,p))$

Then $\mathbb{P}\left(\Delta \geq \underbrace{np + \overset{C}{3}\sqrt{np \log n}}_{d} \right) \to 0$ as $n \to \infty$

Proof. As observed before

$$\mathbb{P}(\Delta \geq d) \leq n\, \mathbb{P}(\underset{\text{Bin}(n-1,p)}{d_v} \geq d) \leq n\, \mathbb{P}(X \geq d)$$

where $X \sim \text{Bin}(n,p)$



Apply Cor 4.2 with $\varepsilon = 3\sqrt{\frac{\log n}{np}} \leq 1$

$$\mathbb{P}(X \geq d) \leq e^{-\frac{\varepsilon^2 np}{4}} = e^{-\frac{9}{4}\log n} = n^{-9/4} = o\left(\frac{1}{n}\right)$$

$\square$