A graph property is something that a graph may or may not have $\leftrightarrow$ a set $\mathcal{P}$ of graphs

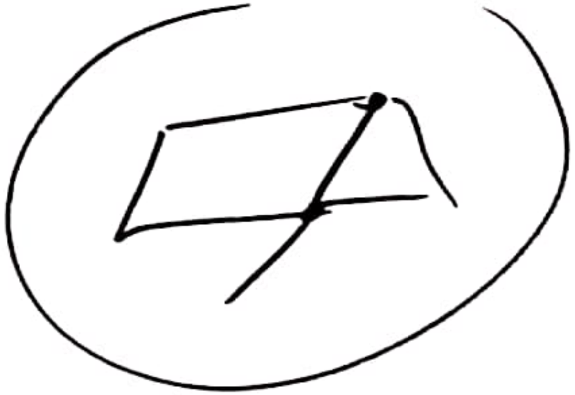
eg 'is connected' ✓ $\leftrightarrow$ $\mathcal{P} = \{\text{connected graphs}\}$

'contains ≥ 1 triangle' ✓

'contains 

' ✓

'has an even # of edges'. ✗

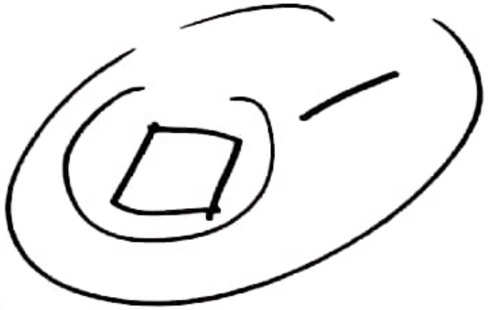


$\mathcal{P}$ is increasing if

- $G \in \mathcal{P} \Rightarrow G+e \in \mathcal{P} \quad \forall e \in G^c$

or

- $G \in \mathcal{P}$, $V(G) = V(G')$, $E(G) \subseteq E(G') \Rightarrow G' \in \mathcal{P}$

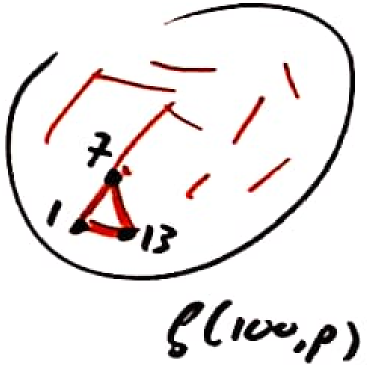


$G = G(n,p)$

Is $\mathbb{P}(G \text{ connected} \mid G \text{ contains} \geq 1\ \Delta) \geq \mathbb{P}(G \text{ is connected})$?

Fix a specific triangle T

Claim $\mathbb{P}(G \text{ conn} \mid G \supseteq T) \geq \mathbb{P}(G \text{ conn})$. ✓



Proof Conditional distribution of $G \mid G \supseteq T$

= unconditional distrib. of $G+T$

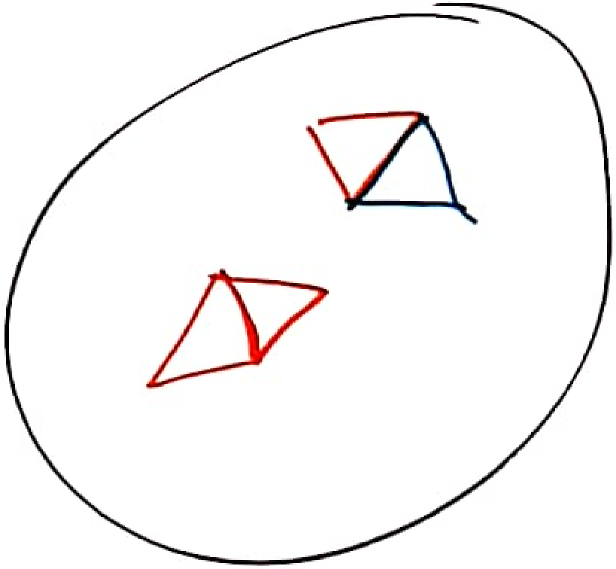
$\mathbb{P}(G \text{ conn} \mid G \supseteq T) = \mathbb{P}(G+T \text{ conn}) \geq \mathbb{P}(G \text{ connected})$

since G conn $\Rightarrow$ $G+T$ connected.

FACT Even if $B_1 \ldots B_k$ are equivalent,

can have $\mathbb{P}(A \mid \cup B_i) < \mathbb{P}(A \mid B_i)$

Let $T_1, \ldots, T_N$ $N = \binom{n}{3}$ list all Δs.



Can check if ≥ 1 present by testing one-by-one. Stop at first 'Yes'

If stop at T_i know : T_i present Δ ☺

$\forall_{j<i}$ T_j NOT present ☹

Let X be a finite set. $E(K_n)$ $\binom{n}{2}$

For $0 \le p \le 1$ let X_p be the random subset with

$\mathbb{P}(i \in X_p) = p$ & these events independent $E(G(n,p))$

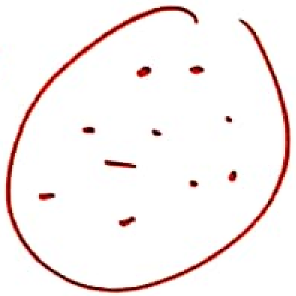
A property of subsets of X is just a collection $\mathcal{A}$ of subsets of X ($\mathcal{A} \subseteq \mathcal{P}(X)$) prop $\mathcal{P}$ of graphs on $[n]$

$\mathcal{A}$ is an increasing property or up-set if

$$A \in \mathcal{A},\ A \subseteq B \subseteq X \Rightarrow B \in \mathcal{A}$$

increasing prop of graphs

[equiv if $A \in \mathcal{A},\ i \in X \Rightarrow A \cup \{i\} \in \mathcal{A}$]



$E(G(n,p))$

$\mathcal{A}$ is decreasing / a down-set if

$$A \in \mathcal{A},\ B \subseteq A \Rightarrow B \in \mathcal{A}$$

Write $\mathbb{P}(\mathcal{A}) = \mathbb{P}_p(\mathcal{A}) = \mathbb{P}_p^X(\mathcal{A})$ for

$$\mathbb{P}(X_p \in \mathcal{A}) = \sum_{A \in \mathcal{A}} p^{|A|}(1-p)^{|X \setminus A|}$$

$\mathbb{P}(G(n,p) \text{ has } \mathcal{P})$

If $p = \frac{1}{2}$ $\mathbb{P}(\mathcal{A}) = \dfrac{|\mathcal{A}|}{2^{|X|}}$

Lemma 6.1 Let X be a finite set, $0 \leq p \leq 1$. If $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}(X)$ are up-sets then

$$\mathbb{P}_p(\mathcal{A} \cap \mathcal{B}) \geq \mathbb{P}_p(\mathcal{A})\, \mathbb{P}_p(\mathcal{B})$$

$$\mathbb{P}(X_p \in \mathcal{A} \cap \mathcal{B}) \geq \mathbb{P}(X_p \in \mathcal{A})\, \mathbb{P}(X_p \in \mathcal{B})$$

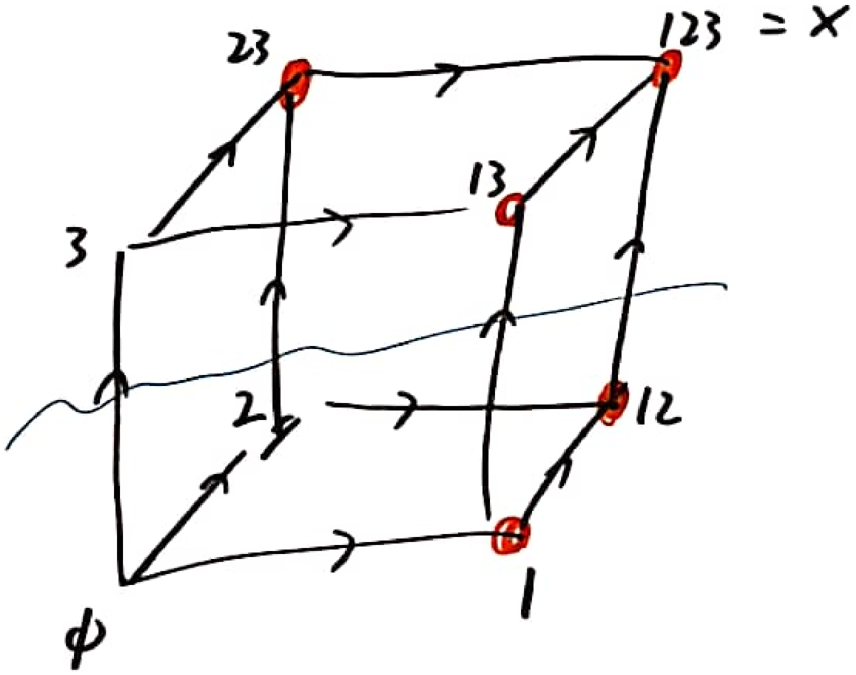
$$\parallel$$

$$\mathbb{P}(X_p \in \mathcal{A} \ \& \ X_p \in \mathcal{B}) \geq$$

$$\mathbb{P}(X_p \in \mathcal{A} \mid X_p \in \mathcal{B}) \geq \mathbb{P}(X_p \in \mathcal{A})$$

"up-sets are +ve correlated"

The discrete cube. Suppose $X = \{1,2,3\}$



$\mathcal{A} = \{1, 12, 13, 23, 123\}$

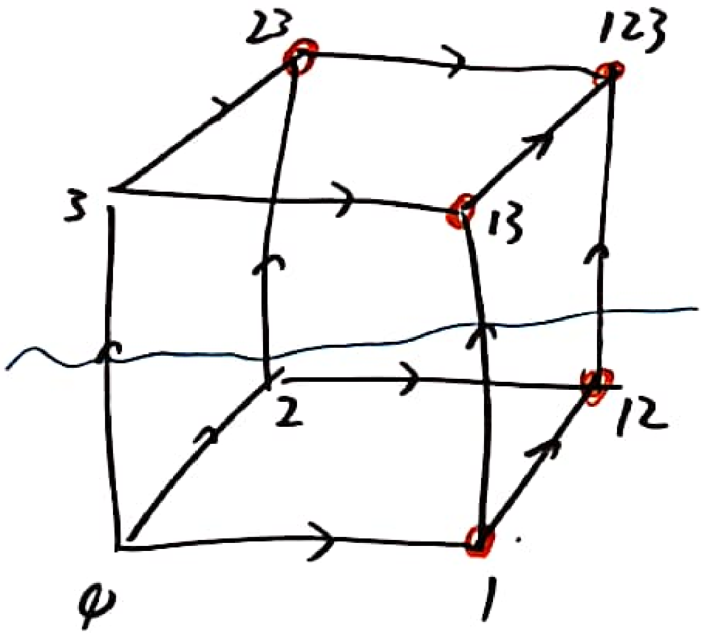
$23 = \{2,3\}$

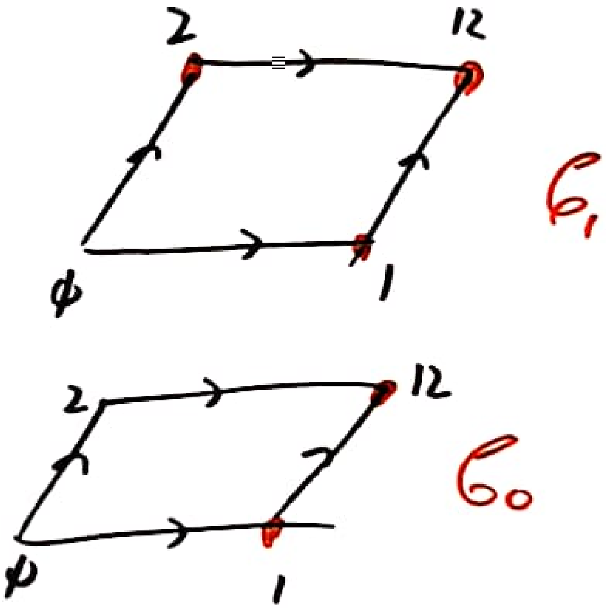
Lemma 6.1 If X finite, $0 \leq p \leq 1$, $\mathcal{A}, \mathcal{B}$ upsets $\mathbb{P}_p^X(\mathcal{A} \cap \mathcal{B}) \geq \mathbb{P}_p^X(\mathcal{A})\mathbb{P}_p^X(\mathcal{B})$

Proof Induction on $n = |X|$.

Base case $n = 0$ ✓ (or $n = 1$ ✓)

Suppose true $|X| < n$ & $|X| = n$. WLOG $X = [n]$





For $\mathcal{C} \subseteq \mathcal{P}(x)$

let $\mathcal{C}_0 = \{ C \in \mathcal{C} : n \notin C \} \subseteq \mathcal{P}([n-1])$

$\mathcal{C}_1 = \{ C \setminus \{n\} : C \in \mathcal{C}, n \in C \} \subseteq \mathcal{P}([n-1])$

Let $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}([n])$ be up-sets

Note $\mathcal{A}_0, \mathcal{A}_1, \mathcal{B}_0, \mathcal{B}_1$ are up-sets. & $\mathcal{A}_0 \subseteq \mathcal{A}_1$, $\mathcal{B}_0 \subseteq \mathcal{B}_1$

Trivially $(\mathcal{A} \cap \mathcal{B})_i = \mathcal{A}_i \cap \mathcal{B}_i$

Also for any $\mathcal{C}$

$$\mathbb{P}_p^{[n]}(\mathcal{C}) = p\,\mathbb{P}_p^{[n-1]}(\mathcal{C}_1) + (1-p)\,\mathbb{P}_p^{[n-1]}(\mathcal{C}_0)$$

Since $X_p = X' \cup X''$ where $X' = [n-1]_p$ $X'' = \{n\}_p$ are independent.

$$\& \quad X_p \in \mathcal{C} \iff \begin{cases} \overset{p}{n \in X''} \ \& \ \overset{\mathbb{P}^{(n-1)}(\mathcal{C}_1)}{X' \in \mathcal{C}_1} \\ \text{or } \overset{1-p}{n \notin X''} \ \& \ \underset{\mathbb{P}^{(n-1)}(\mathcal{C}_0)}{X' \in \mathcal{C}_0} \end{cases}$$

$$\therefore\ \mathbb{P}_p(\mathcal{A}\cap\mathcal{B}) = p\,\mathbb{P}((\mathcal{A}\cap\mathcal{B})_1) + (1-p)\,\mathbb{P}((\mathcal{A}\cap\mathcal{B})_0)$$

$$a_0 = \mathbb{P}(\mathcal{A}_0)$$

$$= p\,\mathbb{P}^{[n-1]}(\mathcal{A}_1\cap\mathcal{B}_1) + (1-p)\,\mathbb{P}(\mathcal{A}_0\cap\mathcal{B}_0)$$

$$\geq p a_1 b_1 + (1-p) a_0 b_0 = x$$

$$\mathbb{P}(\mathcal{A})\,\mathbb{P}(\mathcal{B}) = (p a_1 + (1-p) a_0)(p b_1 + (1-p) b_0) = y.$$

STP $x \geq y$.

But $x - y = \overset{p(1-p)}{(p-p^2)}a_1 b_1 - p(1-p)a_0 b_1 - p(1-p)a_1 b_0 + \overset{(1-p)p}{((1-p)-(1-p)^2)}a_0 b_0$

$$= p(1-p)(a_1 b_1 - a_0 b_1 - a_1 b_0 + a_0 b_0)$$

$$= p(1-p)\underbrace{(a_1 - a_0)}_{\geq 0}\underbrace{(b_1 - b_0)}_{\geq 0} \geq 0 \quad \square$$

$\mathcal{A}_0 \subseteq \mathcal{A}_1$

$\mathbb{P}(\mathcal{A}_0) \leq \mathbb{P}(\mathcal{A}_1)$

$a_0 \leq a_1$

Cor 6.2 If $\mathcal{U}$ is an up-set and $\mathcal{D}_1, \mathcal{D}_2$ are down-sets then

$$\mathbb{P}_p(\mathcal{U} \cap \mathcal{D}_1) \leq \mathbb{P}_p(\mathcal{U})\mathbb{P}_p(\mathcal{D}_1)$$

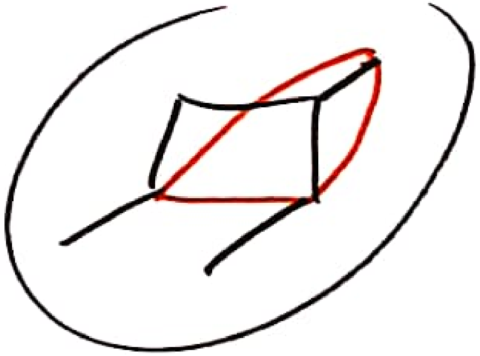
$$\mathbb{P}_p(\mathcal{D}_1 \cap \mathcal{D}_2) \geqslant \mathbb{P}_p(\mathcal{D}_1)\mathbb{P}_p(\mathcal{D}_2).$$

Pf Exercise. Show $\mathcal{D}_i^c$ is an up-set. □

$\mathbb{P}($ $\exists$ spanning tree $)$ & $\exists$ triangle

$\mathbb{P}\big($ $\exists$ spanning tree & a <u>disjoint</u> triangle $\big)$

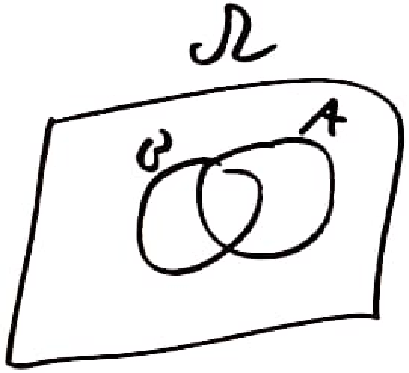
$\leq \mathbb{P}(\qquad\qquad)\, \mathbb{P}(\qquad\qquad)$



Van den Berg & Kesten

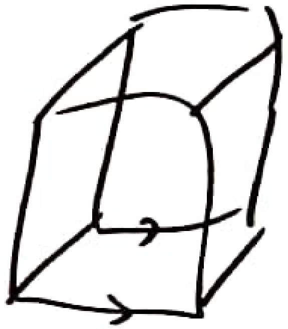
Reimer

Convention set	X	Ω
elements	i, j, n	ω
subsets	A, B, X	A, B
sets of subsets	$\mathcal{A}, \mathcal{B}$	$\mathcal{F}$



$$\Omega = \mathcal{P}(X)$$

$$\mathcal{P}(E(K_n))$$



$X_p \subseteq X$

Setting for Janson : X a ground set $E(K_n)$

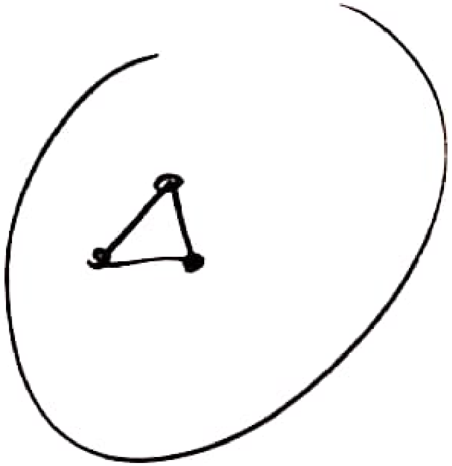
X_p random subset $E(G(n,p))$

For $E \subseteq X$ let $A = \{X_p \supseteq E\}$

(in 'Harris' notation $\mathcal{A} = \{Y \subseteq X : E \subseteq Y\}$)

This is an up-set.

Called a principal up-set.



A = this Δ present

Let $E_1, \dots, E_k \subseteq X$, let $A_i = \{X_p \supseteq E_i\}$

Let Z = # of A_i that hold.

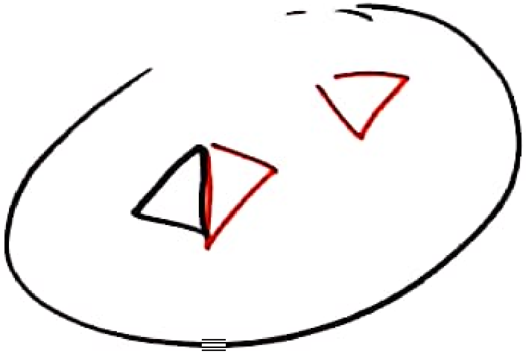
Aim prove $\mathbb{P}(Z=0) \leq$ ——— $e^{-\mu + \Delta/2}$

in terms of $\mu = \cancel{\Delta}\, \mathbb{E}[Z] = \sum \mathbb{P}(A_i)$ & ~

$$\Delta = \sum_i \sum_{j \sim i} \mathbb{P}(A_i \cap A_j)$$

where $i \sim j$ if $i \neq j$ & – A_i & A_j are dependent

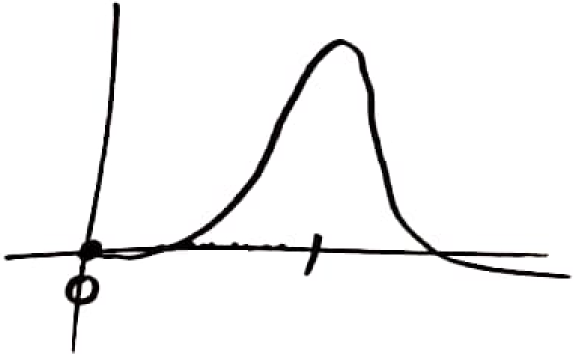
\ $E_i \cap E_j \neq \phi$



(NB as in Chapter 2 $\mathrm{Var}[Z] \leq \mu + \Delta$

(LL $\mathbb{P}(Z=0) \geq$.

Here $\mathbb{P}(Z=0) \leq$.)



$$e^{-\mu+\Delta/2}$$

N.B. If indp $\mathbb{P}(Z=0) = \mathbb{P}(\cap A_i^c) = \prod \mathbb{P}(A_i^c)$

$$= \prod (1-\mathbb{P}(A_i)) \approx \prod e^{-\mathbb{P}(A_i)} = e^{-\mu}$$

$1-x \approx e^{-x}$

Remark: each A_i is an up-set. Each A_i^c is a down-set

Follows from Harris $\mathbb{P}(\cap A_i^c) \geq \prod \mathbb{P}(A_i^c)$

✓ $\approx e^{-\mu}$

Thm 6.3 (Janson's Ineq.) In setting just described

$$\mathbb{P}(Z=0) \leq e^{-\mu+\Delta/2}$$

Proof $\mathbb{P}(Z=0) = \mathbb{P}\left(\bigcap_{i=1}^{k} A_i^c\right) = \mathbb{P}(A_1^c)\,\mathbb{P}(A_2^c | A_1^c) \cdots\cdots$

Let $r_i = \mathbb{P}(A_i \mid A_1^c \cap \cdots \cap A_{i-1}^c)$

So $\mathbb{P}(Z=0) = \prod_{i=1}^{k}(1-r_i) \leq \prod e^{-r_i} = e^{-\sum r_i}$ $\quad 1-x \leq e^{-x}$

Aim: show $\quad r_i \geqslant \mathbb{P}(A_i) -$ little bit

$$r_i = \mathbb{P}(A_i \mid D_0 \cap D_1) = \frac{\mathbb{P}(A_i \cap D_0 \cap D_1)}{\mathbb{P}(D_0 \cap D_1)} \geq \frac{\mathbb{P}(A_i \cap D_0 \cap D_1)}{\mathbb{P}(D_0)}$$

$$= \mathbb{P}(A_i \cap D_1 \mid D_0) = \underset{\mathbb{P}(A_i)}{\underset{\shortparallel}{\mathbb{P}(A_i \mid D_0)}} - \underbrace{\mathbb{P}(\underbrace{A_i \cap D_1^c}_{\text{up-set}} \mid D_0)}_{\leq\, \mathbb{P}(A_i \cap D_1^c)} \quad \text{HARRIS}$$

Now A_i is an up-set. D_0, D_1 are down-sets $\therefore$ D_1^c is an up-set

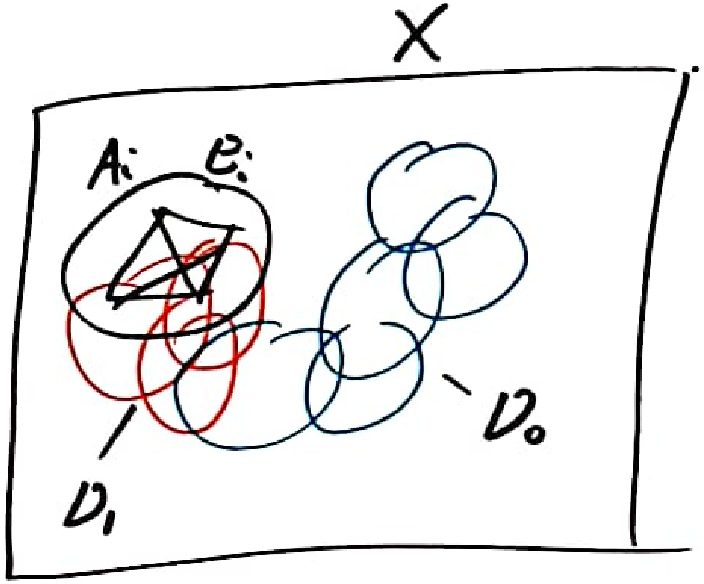
$$\geq \mathbb{P}(A_i) - \mathbb{P}(A_i \cap D_1^c)$$

Fix i.

Let $D_0 = \bigcap_{\substack{j<i \\ j\not\sim i}} A_j^c$

~~D_1~~

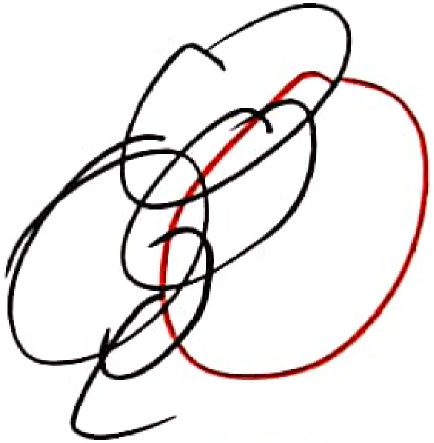
$D_1 = \bigcap_{\substack{j<i \\ j\sim i}} A_j^c$



N.B whether D_0 holds is determined by presence/absence in X_p of elements in $\bigcup_{\substack{j<i \\ j\not\sim i}} E_j$ which is by defn disjoint for E_i

$\therefore$ A_i & D_0 INDEP.

$$D_i^c = \left(\bigcap_{\substack{j<i \\ j\sim i}} A_j^c\right)^c = \bigcup_{\substack{j<i \\ j\sim i}} A_j$$



$$A_i \cap D_i^c = \bigcup_{\substack{j<i \\ j\sim i}} A_i \cap A_j$$

$A_1 \cap A_3$

$i=1, j=3$

$i=3, j=1$

$$\mathbb{P}\left(\qquad \right) \le \sum_{\substack{j<i \\ j\sim i}} \mathbb{P}\left(\quad \right)$$

$\Delta = \sum_i \sum_{j\sim i} \mathbb{P}(A_i \cap A_j)$

$$\therefore \sum r_i \geq \sum \mathbb{P}(A_i) - \sum_i \sum_{\substack{\boxed{j<i} \\ j\sim i}} \mathbb{P}(A_i \cap A_j)$$

$$= \mu - \Delta/2$$

$$\mathbb{P}(Z=0) \le e^{-\sum r_i} \le e^{-\mu + \Delta/2}$$

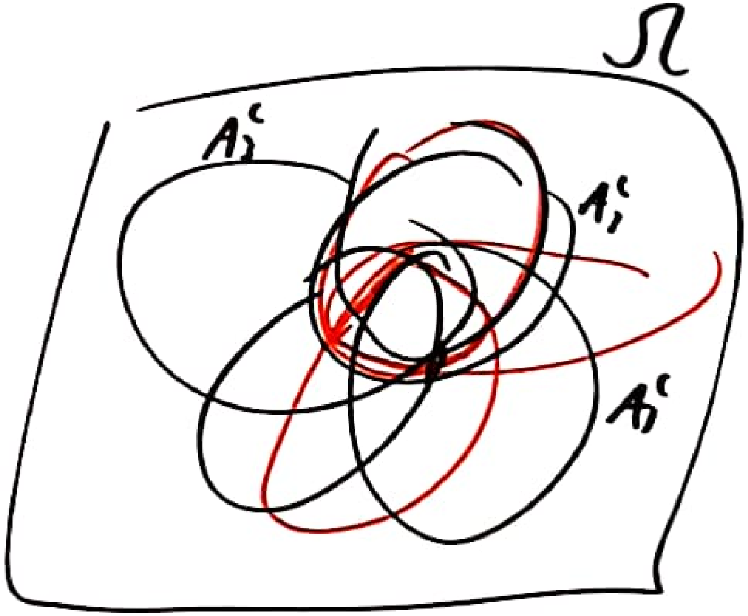
□

Thm 6.4 In setting of Thm 6.3, if $\Delta \geq \mu$, $\mathbb{P}(Z=0) \leq e^{-\frac{\mu^2}{2\Delta}}$

$\mathbb{P}(Z=0) \leq e^{-\mu+\Delta/2}$

Proof

$(Z=0 = \bigcap A_i^c)$



For any subset $S \subseteq [k]$

$$\mathbb{P}(Z=0) = \mathbb{P}\left(\bigcap_{i=1}^{k} A_i^c\right)$$

$$\leq \mathbb{P}\left(\bigcap_{i\in S} A_i^c\right) \leq \exp\left(-\mu_S + \Delta_{S/2}\right)$$ by Thm 6.3

where

$$\mu_S = \sum_{i\in S} \mathbb{P}(A_i) \qquad \Delta_S = \sum_{i\in S} \sum_{\substack{j\in S \\ j\sim i}} \mathbb{P}(A_i \cap A_j)$$

$\Omega = \mathcal{P}(X)$

X_p

Idea choose S so $\mu_S - \Delta_S/2$ big.

$\tilde{\Omega} = \mathcal{P}([k])$

S

Fix $0 \le r \le 1$. Let S be a random subset of $[k]$ where each $i \in [k]$ is included with prob. r indep. of others.

$$\text{Then} \quad \mu_S = \sum_{i=1}^{k} \underbrace{\mathbb{1}_{\{i\in S\}}}_{\text{RV}} \underbrace{\mathbb{P}(A_i)}_{\text{constant}} \qquad \Delta_S = \sum_{i} \sum_{j\sim i} \mathbb{I}_{\{i,j\in S\}} \mathbb{P}(A_i \cap A_j)$$

$$\therefore \quad \tilde{\mathbb{E}}\mu_S = \sum_{i=1}^{k} \mathbb{P}(A_i)\, \widehat{\mathbb{E}}(\mathbb{1}_{\{i\in S\}}) = \sum_{i=1}^{k} \mathbb{P}(A_i)\, \underbrace{\widehat{\mathbb{P}}(i\in S)}_{r} = r\sum \mathbb{P}(A$$

$$= r\mu$$

$$\tilde{\mathbb{E}}(\Delta_S) = \sum_i \sum_{j \sim i} \mathbb{P}(A_i \cap A_j) \, \underline{\tilde{\mathbb{E}}[I_{\{i,j \in S\}}]}$$

$$\tilde{\mathbb{P}}(i \in S, j \in S) = r^2$$

$$= r^2 \Delta$$

$$\therefore \tilde{\mathbb{E}}[\mu_S - \Delta_S/2] = r\mu - r^2\Delta/2$$

Since $\tilde{\mathbb{P}}(Y \geq \mathbb{E}Y) > 0$ $\exists\, S$ s.t. for this S $\mu_S - \Delta_S/2 \geq r\mu - r^2\Delta/2$

$$\therefore \mathbb{P}(Z = 0) \leq e^{-r\mu + r^2\Delta/2}$$

Take $r = \frac{\mu}{\Delta} \leq 1$

$$e^{-\frac{\mu^2}{2\Delta}}$$

$\square$

Cor 6.5 In setting of Thm 6.3

$$\mathbb{P}(Z=0) \leq \exp\left(-\min\left\{\frac{\mu}{2}, \frac{\mu^2}{2\Delta}\right\}\right) \leq e^{-L/2}$$

Proof If $\Delta < \mu$ use Thm 6.3 get $e^{-\mu+\Delta/2} \leq e^{-\mu/2}$

$\Delta \geqslant \mu$ 6.4 □

Recall $\operatorname{Var} Z \leq \mu + \Delta$

Chebyshev gives $\mathbb{P}(Z=0) \leq \frac{\operatorname{Var} Z}{\mu^2} \leq \frac{1}{\mu} + \frac{\Delta}{\mu^2} \leq \frac{2}{L}$

Suppose $\mu, \frac{\mu^2}{\Delta} \geqslant L$