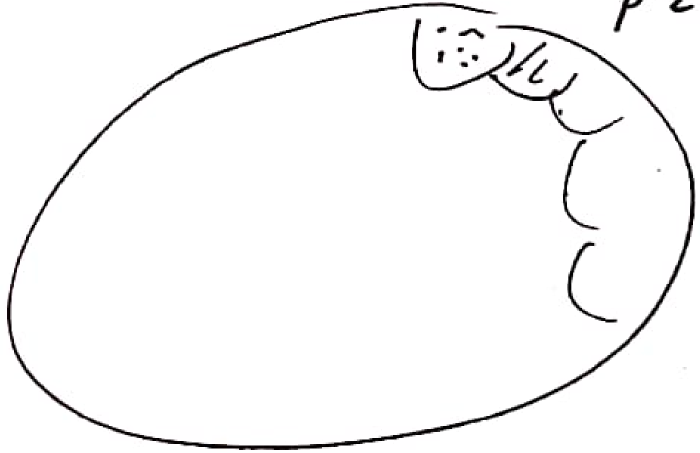
Clique number of $G(n,p)$.

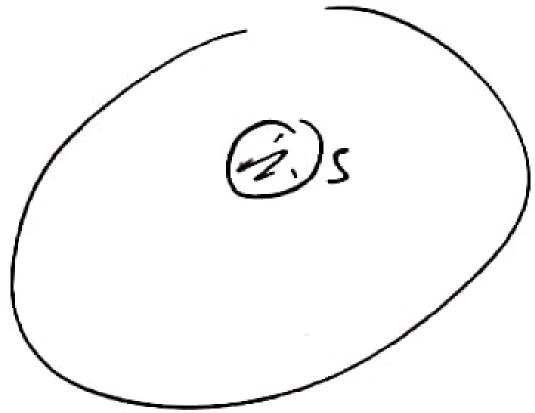
Throughout $0<p<1$ is constant.

But $k = k(n)$ is not.



Let X_k ($= X_{n,k}$) $=$ # k-cliques (copies of K_k) in $G(n,p)$

$$\mu_k := \mu_{n,k} = \mathbb{E}[X_k] = \binom{n}{k} p^{\binom{k}{2}}$$



Vaguely

$$\mu_k \approx n^k p^{k^2/2}$$

$$\log \mu_k \approx k \log n - \frac{k^2}{2} \underbrace{\log(1/p)}_{>0}$$

$|S| = k$ (

$$\binom{k}{2} = \frac{k(k-1)}{2}$$

$$\mu_k = \binom{n}{k} p^{\binom{k}{2}}$$

$$\mu_0 = 1$$

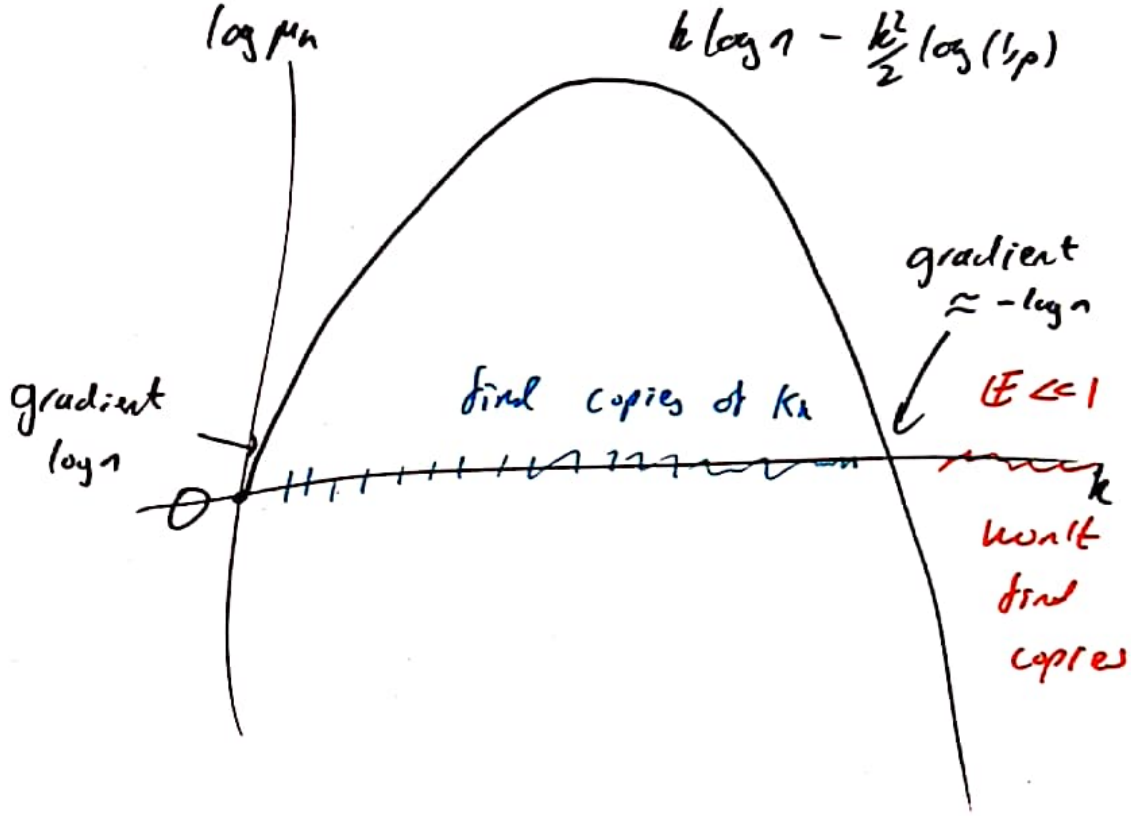
Note

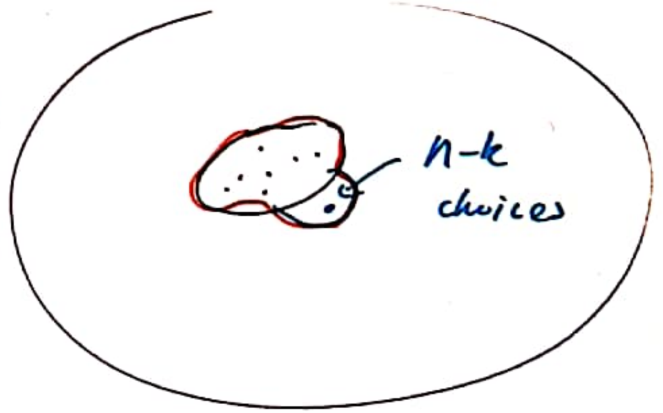
$$\frac{\mu_{k+1}}{\mu_k} = \frac{\binom{n}{k+1}}{\binom{n}{k}} p^{\binom{k+1}{2} - \binom{k}{2}}$$

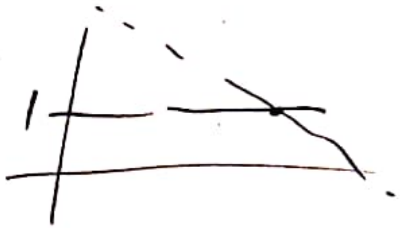
$$= \frac{n-k}{k+1} p^k$$

which is decreasing

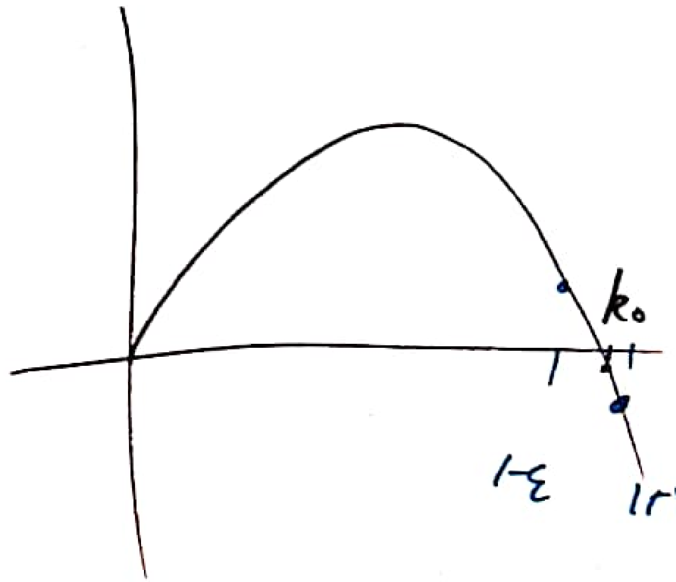
$\therefore$ ratio ≥ 1 up to some point, then < 1







$\therefore$ μ_k increases, then decreases



Define $k_0 = k_0(n,p) =$

$\min\{k : \mu_k < 1\}$

<u>Lemma 7.1</u> $k_0 \sim 2\log_{1/p}(n) = \dfrac{2\log n}{\log(1/p)}$

<u>Pf</u> By standard bounds on $\binom{n}{k}$

$$\left(\frac{n}{k}\right)^{\cancel{k}} p^{\frac{\cancel{k}(k-1)}{2}} \le \mu_k^{1/k} \le \left(\frac{en}{k}\right)^{\cancel{k}} p^{\frac{\cancel{k}(k-1)}{2}}$$

(for $x > 0$

$x^k > 1 \iff x > 1$

$$\therefore \quad \mu_k^{1/k} = \Theta\left(\frac{n}{k} p^{\frac{k-1}{2}}\right) = \Theta\left(\frac{n}{k} p^{\frac{k}{2}}\right)$$

By definition $\left(\frac{1}{p}\right)^{\log_{1/p} n} = n$ $\qquad \frac{n}{k} \approx n \quad p^{\frac{k}{2}}$

$$p^{\log_{1/p} n} = n^{-1}$$

$$p^{c \log_{1/p} n} = n^{-c}$$

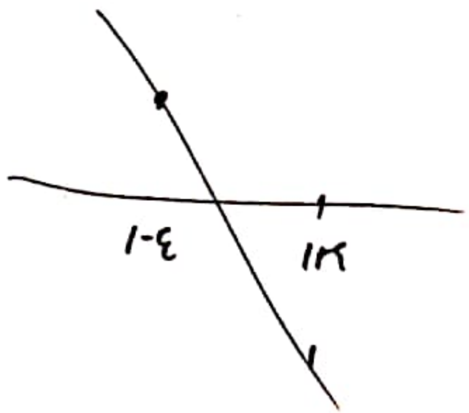
If $k \geq (1+\varepsilon)\, 2 \log_{1/p} n$ $\qquad p^{k/2} \leq n^{-1-\varepsilon} \Rightarrow \frac{n}{k} p^{\frac{k}{2}} \leq n \cdot n^{-1-\varepsilon} \leq n^{-\varepsilon}$

< 1

$k \leq (1-\varepsilon) \sim$ $\qquad p^{\frac{k}{2}} \geq n^{-1+\varepsilon}$

$$\frac{n}{k} p^{\frac{k}{2}} \geq \frac{n \cdot n^{-1+\varepsilon}}{C \log n}$$

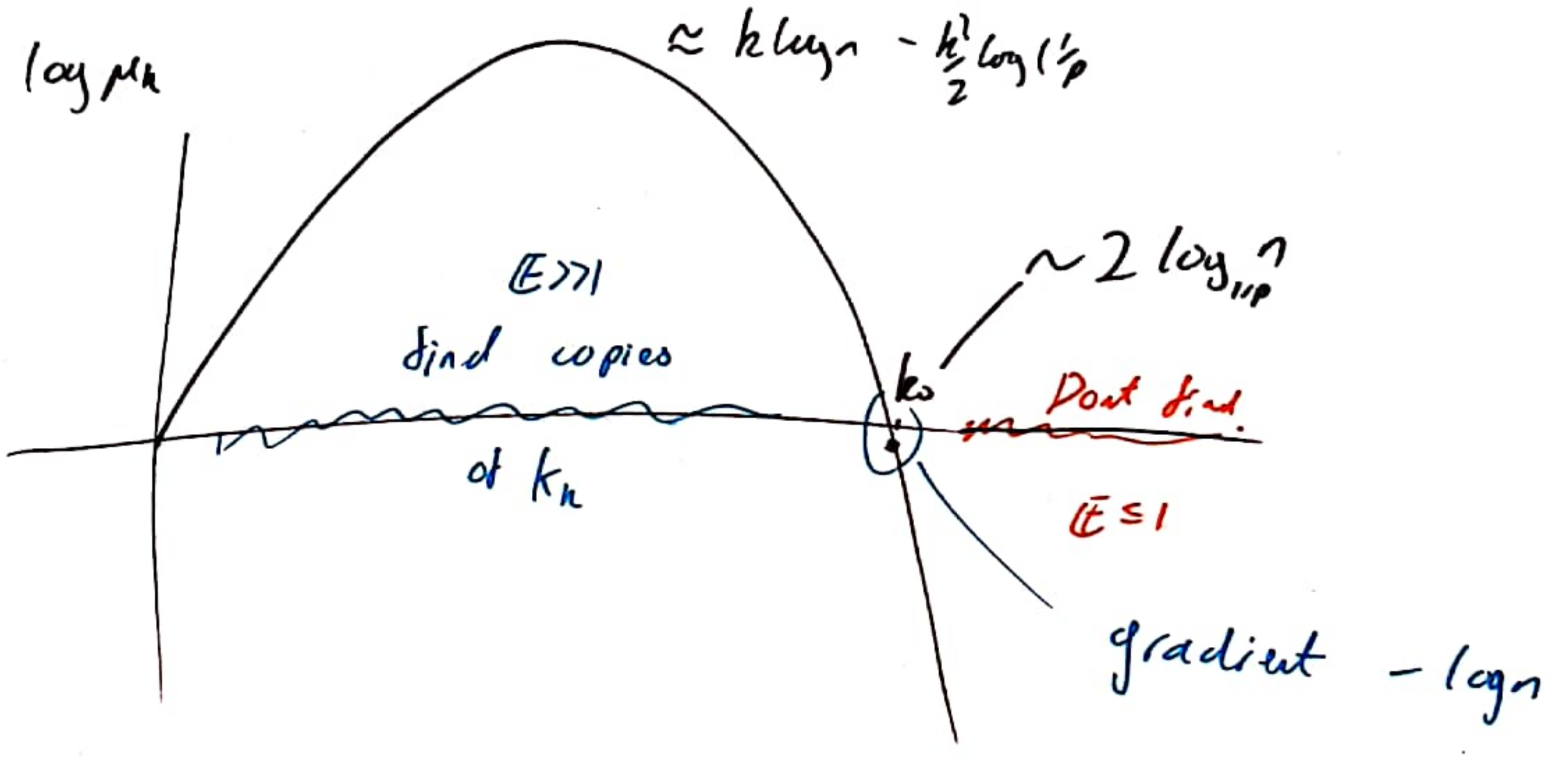
> 1 if n large



$\therefore$ if n large

$(1-\varepsilon) \leq k_0 \leq (1+\varepsilon) \sim$

$\square$



For $k \sim k_0 = 2\log_{1/p} n$

$$\left(\frac{1}{p}\right)^k = n^{2+o(1)}$$

$$\log n = n^{o(1)}$$

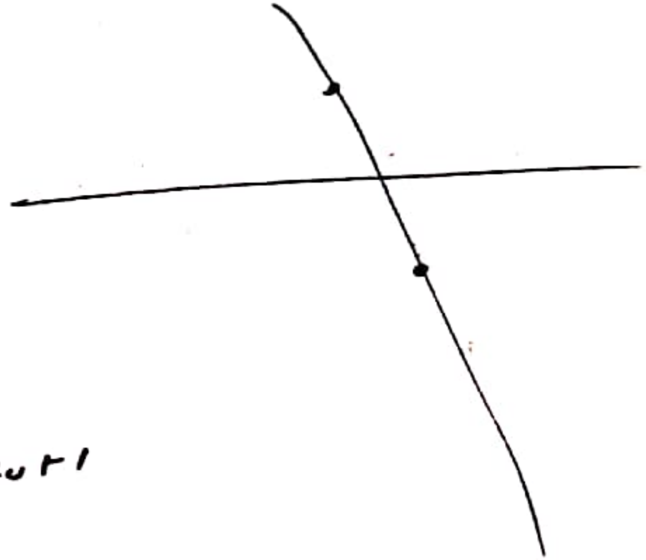
$$\therefore \quad \frac{\mu_{k+1}}{\mu_k} = \frac{n-k}{k+1} p^k = \frac{n}{n^{o(1)}} n^{-2+o(1)} = n^{-1+o(1)}$$

Lemma 7.2 $\quad \mathbb{P}(\omega(G(n,p)) > k_0) \to 0$

$\Updownarrow$

Pf $\quad \Leftrightarrow X_{k_0+1} \geq 1$

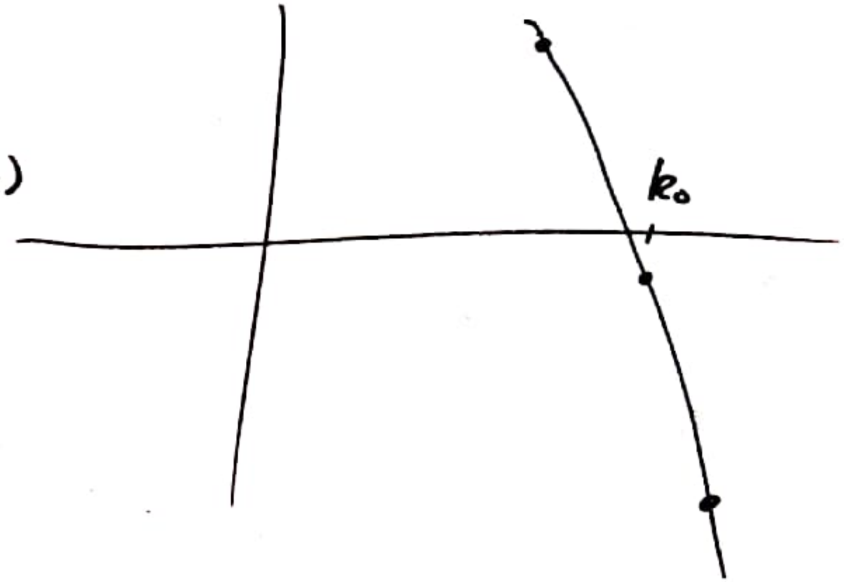
$\therefore \mathbb{P}(\quad\quad) \leq \mathbb{E}[X_{k_0+1}] = \mu_{k_0+1}$



By defn $\quad \mu_{k_0} < 1$

$$\therefore \quad \mu_{k_0+1} = \mu_{k_0} \cdot \frac{\mu_{k_0+1}}{\mu_{k_0}} \leq n^{-1+o(1)}$$

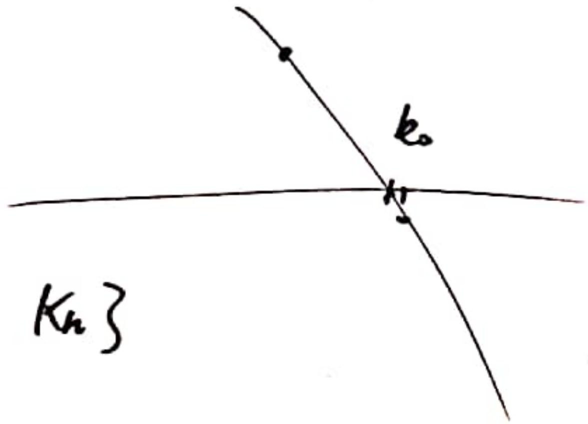
$n^{-1+o(1)}$



$\to 0$

□

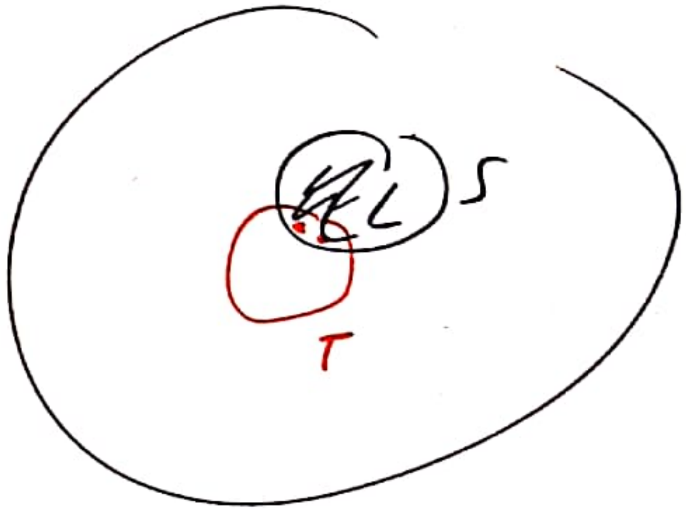
Consider some k near k_0, X_k



$$X_k = \sum_{S:|S|=k} 1_{A_S} \qquad A_S : \{S \text{ spans a } K_k\}$$

Write $S \sim T$ if $S \neq T$ & A_S, A_T dependendent

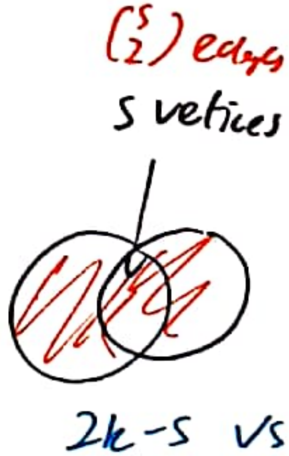
ie. if $2 \leq |S \cap T| \leq k-1$

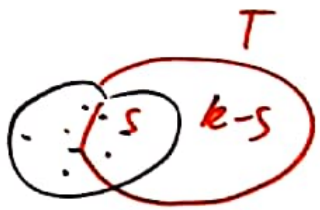


Define

$$\Delta_k = \sum_S \sum_{T \sim S} \mathbb{P}(A_S \cap A_T)$$

$$= \sum_{s=2}^{k-1} \binom{n}{k}\binom{k}{s}\binom{n-k}{k-s} p^{2\binom{k}{2}-\binom{s}{2}}$$





$$\therefore \frac{\Delta_k}{\mu_k^2} = \sum_{s=2}^{k-1} \alpha_s \quad , \quad \alpha_s = \frac{\binom{k}{s}\binom{n-k}{k-s}}{\binom{n}{k}} \left(\frac{1}{p}\right)^{\binom{s}{2}}$$

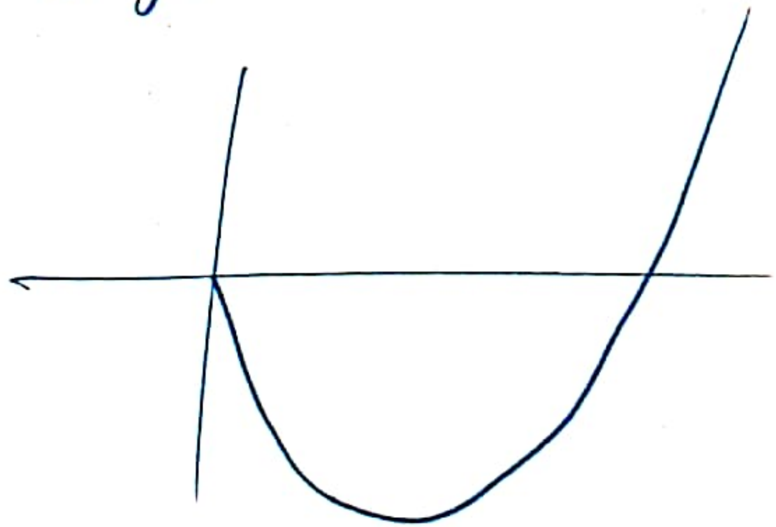
NB sum $\leq k \max_s \alpha_s = n^{o(1)} \max_s \alpha_s$.

$$\Delta_k \approx n^{2k-s} p^{2\frac{k^2}{2} - \frac{s^2}{2}}$$

$$\alpha_s \approx n^{-s} \left(\frac{1}{p}\right)^{s^2/2} \qquad \log \alpha_s \approx \frac{s^2}{2} \log(1/p) - s \log n$$

Idea: show $\exists s_0$ s.t. α_s decreases $s \leq s_0$

then increases



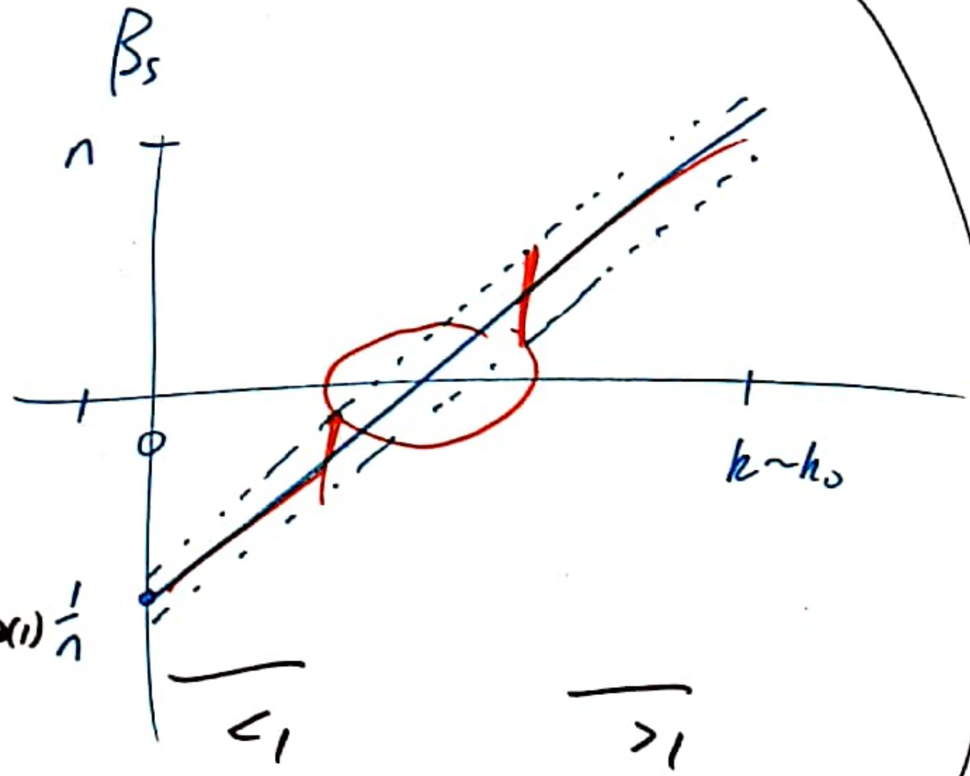
$$\Rightarrow \max_{2 \leq s \leq k-1} \alpha_s = \max\{\alpha_2, \alpha_{k-1}\}$$

$\boxed{k \sim k_0}$

Let $\beta_s = \frac{\alpha_{s+1}}{\alpha_s} = \frac{k-s}{s+1} \frac{k-s}{n-2k+s+1} \left(\frac{1}{p}\right)^s$

$= n^{-1+o(1)} \left(\frac{1}{p}\right)^s$

Idea $\beta_s \le 1$ up to s_0
>1 beyond s_0



For $s < k/4$ $\left(\frac{1}{p}\right)^s < \left(\frac{1}{p}\right)^{k/4} = n^{\frac{1}{2}+o(1)}$

$\therefore \beta_s < 1$

$s > 3k/4$ $\left(\frac{1}{p}\right)^s > \left(\frac{1}{p}\right)^{3k/4} = n^{\frac{3}{2}+o(1)}$

$\therefore \beta_s > 1$

For $k/4 < s < 3k/4$ $\frac{\beta_{s+1}}{\beta_s} \sim \frac{1}{p} > 1$ $\therefore$ done : provet

Lemma 7.3 Suppose $k \sim k_0$. Then $\dfrac{\Delta_k}{\mu_k^2} \leq \max\left\{ n^{-2+o(1)}, \dfrac{n^{-1+o(1)}}{\mu_k} \right\}$.

In particular, if $\mu_k \to \infty$, $\Delta_k = o(\mu_k^2)$

Proof

$$\frac{\Delta_k}{\mu_k^2} = \sum_{s=2}^{k-1} \alpha_s = n^{o(1)} \max_{2 \leq s \leq k-1} \alpha_s = n^{o(1)} \max\{\alpha_2, \alpha_{k-1}\}$$

$$\alpha_s = \frac{\binom{k}{s}\binom{n-k}{k-s}}{\binom{n}{k}} p^{-\binom{s}{2}} \qquad \alpha_0 = \frac{\binom{n-k}{k}}{\binom{n}{k}} \leq 1$$

$$\alpha_k = \frac{1}{\mu_k}$$

$$\beta_s = n^{-1+o(1)} \left(\frac{1}{p}\right)^s. \qquad \alpha_2 = \alpha_0 \beta_0 \beta_1 = \alpha_0 n^{-2+o(1)} \leq n^{-2+o(1)}$$

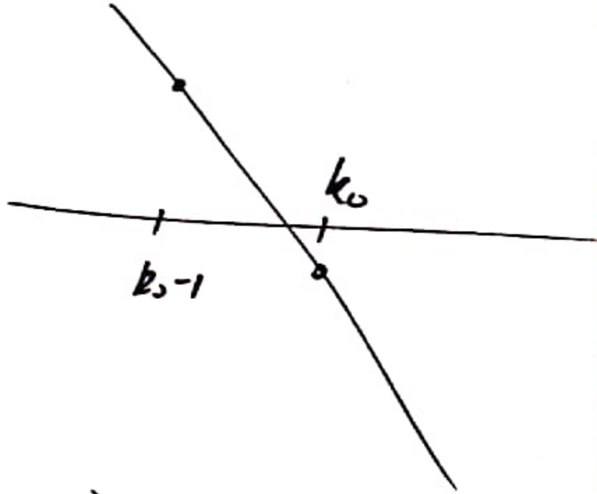
$$\alpha_{k-1} = \alpha_k \beta_k^{-1} = \alpha_k \left(n^{-1} n^{2+o(1)}\right)^{-1} = n^{-1+o(1)} / \mu_k \qquad \square$$

Theorem 7.4 Fix $0 < p < 1$. Define $k_0 = k_0(n,p)$ as before

Then
$$\mathbb{P}\left(k_0 - 2 \leq \omega(G(n,p)) \leq k_0 \right) \to 1.$$

Pf Lemma 7.2 say $\mathbb{P}(\omega > k_0) \to 0$.

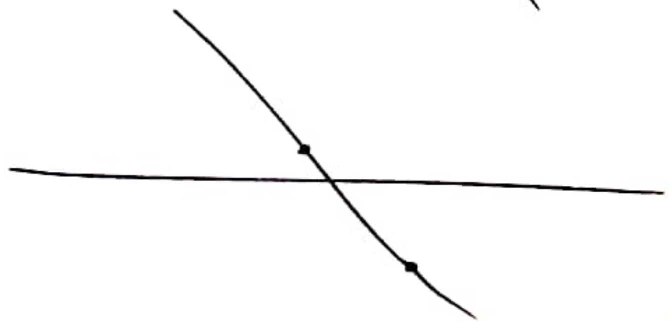
Let $k = k_0 - 2$. STP $\mathbb{P}(X_k > 0) \to 1$



By defn $\mu_{k_0-1} \geq 1$

$$\therefore \mu_k = \mu_{k_0-2} \geq n^{1-o(1)} \to \infty.$$

By Lemma 7.3 $\dfrac{\Delta_k}{\mu_k^2} \to 0$ $\therefore$ done by Cor 2.4. $\square$



Thm 7.5 Same assumptions: $\mathbb{P}(\omega(G) < k_0 - 3) \le e^{-n^{2-o(1)}}$.

Pf $\mu_{k_0-1} \geq 1 \quad \therefore \quad \mu_k \ge n^{2-o(1)}$ $\qquad k = k_0 - 3$

$$\therefore \frac{\Delta_k}{\mu_k^2} \le \max\left\{ n^{-2+o(1)}, \frac{n^{-1+o(1)}}{\mu_k} \right\} \le n^{-2+o(1)}$$

$$\therefore \frac{\mu_k^2}{\Delta_k} \geq n^{2-o(1)}$$

By Cor 6.5 (Janson)

$$\mathbb{P}(X_k = 0) \le e^{-\min\left\{\frac{\mu_k}{2}, \frac{\mu_k^2}{2\Delta_k}\right\}} = e^{-n^{2-o(1)}}$$

$\square$

$$\mathbb{P}(e(G(n,p)) = 0) = (1-p)^{\binom{n}{2}} = e^{-(c+o(1))n^2}$$

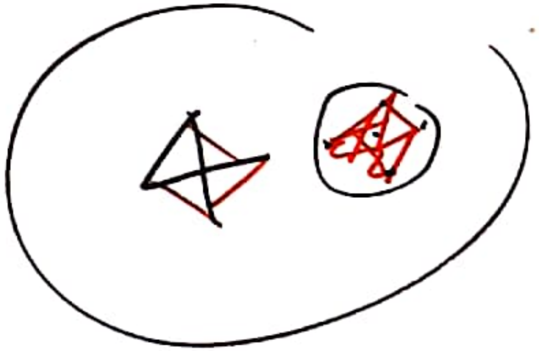
Theorem 7.6 (Bollobás, 1988) Let $0<p<1$ be constant.

For any $\varepsilon>0$ whp

$$(1-\varepsilon)\frac{n}{2\log_b n} \le \chi(G(n,p)) \le (1+\varepsilon)\frac{n}{2\log_b n}$$

where $b=\frac{1}{1-p}$.

Pf Recall $\chi(G) \geq \frac{n}{\alpha(G)} = \frac{n}{\omega(G^c)}$



Let $G = G(n,p)$. Then $G^c \sim G(n,1-p)$

By Thm 7.4 whp $\alpha(G) = \omega(G^c) \le k_0(n,1-p) \sim 2\log_b n$

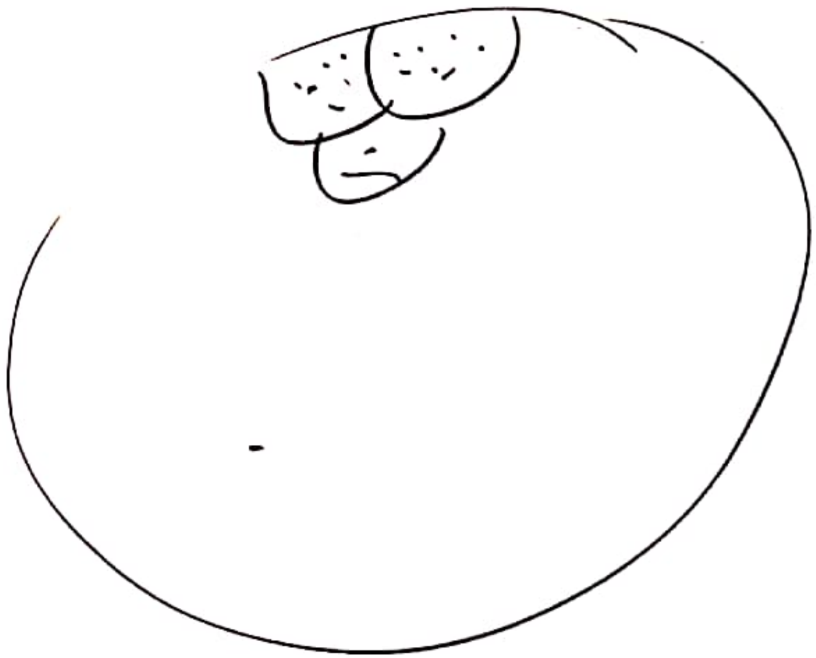
$\therefore$ whp $\alpha(G) \le (1+\varepsilon)2\log_b n \Rightarrow \chi \geq \frac{n}{\alpha(G)} \geq \frac{1}{1+\varepsilon} 2\log_b n$

Idea $\alpha(G) \approx 2\log_b n$

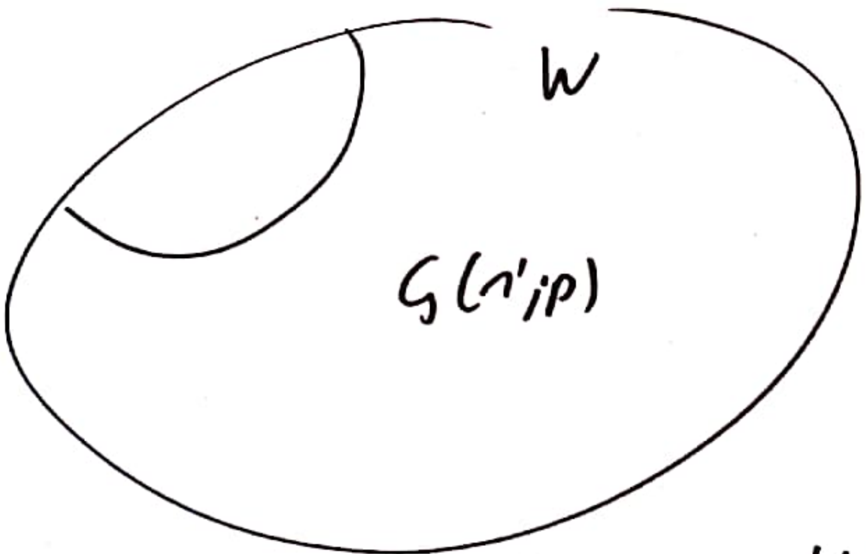
For n' similar to n ($n' = n^{1+o(1)}$)

$\alpha(G(n',p)) \sim 2\log_b n' \sim 2\log_b n$

whp



$G(n',p)$



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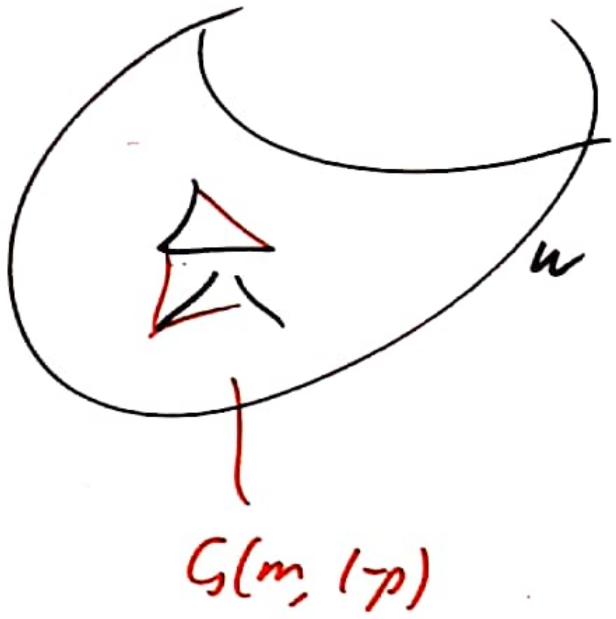
Let $m = \left\lfloor \frac{n}{(\log n)^2} \right\rfloor$. $= n^{1+o(1)}$

For each set W of m vxs of G let E_W be event that $G[W]$ contains an indp. set of size $\geq k = k_0(m, 1-p) - 3$.

By Thm 7.5 applied to $G[W]^c$



$$\mathbb{P}(E_W^c) = \mathbb{P}(\omega(G(m, 1-p)) < k_0(m, 1-p) - 3)$$

$$\leq e^{-m^{2+o(1)}} = e^{-n^{2+o(1)}}$$

By union bound $\quad E = \bigcap_{W : |W| = m} E_W$

$$\mathbb{P}(E^c) \leq \sum_W \mathbb{P}(E_W^c) \leq \binom{n}{m} e^{-n^{2+o(1)}} \leq 2^n e^{-n^{2+o(1)}} \to 0$$

So whp E holds.

Claim: whenever E holds $\chi(G) \leq (1+\varepsilon)\dfrac{n}{2\log_b n}$ (n large enough)

<u>Proof</u> note $k \sim k_0(m, 1-p) \sim 2\log_b m \sim 2\log_b n$

When E holds any set of m vxs contains indp set size k

Remove such sets until $\leq m$ vxs left.

Each removed set gets a colour, remaining vxs their own colours

$$\# \text{colours} \leq \frac{n}{k} + m = (1+o(1))\frac{n}{2\log_b n} + o\left(\frac{n}{\log n}\right) \sim \frac{n}{2\log_b n} \quad \square$$