C8.2: Stochastic analysis and PDEs
Problem sheet 1

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The questions on this sheet are divided into two sections. Those in the first section are compulsory and should be handed in for marking. Those in the second are extra practice questions and should not *be handed in.*

Section 1 (Compulsory)

1. Give an example of two stochastic processes X and Y with X continuous, Y discontinuous, but with X and Y having the same finite dimensional distributions.

2. Suppose that a continuous time Markov chain X can be in any one of three states, which we label $\{A, B, C\}$. The Q-matrix is given by

$$Q = \begin{pmatrix} -3 & 1 & 2 \\ 0 & -2 & 2 \\ 2 & 1 & -3 \end{pmatrix}.$$

   Describe the dynamics of the chain. Find the AA entry in the corresponding resolvent. What is $\mathbb{P}_A[X(t) = A]$?

3. Let T_t be a strongly continuous contraction semigroup with resolvents R_λ, $\lambda > 0$. Verify the *resolvent equation*

$$R_\lambda - R_\mu = (\mu - \lambda) R_\lambda R_\mu$$

   and deduce that the operators R_λ commute.

4. For S and T independent exponential random variables with rates λ and μ respectively, where $\lambda \neq \mu$, find the density of $S + T$ and hence give a probabilistic interpretation of the resolvent equation.

5. Let C be a Banach space. Suppose that the linear operator T on C is a strict contraction (i.e. $\|T\| < 1$ where $\|\cdot\|$ is the operator norm). Show that $(I - T)^{-1}$ exists as a bounded linear operator and

$$(I - T)^{-1} = \sum_{j=0}^{\infty} T^j.$$

   Now take R_λ and R_μ as in Question 3 and show that if $|\lambda - \mu| < \|R_\lambda\|^{-1}$, then

$$(\mu - A)^{-1} = \sum_{n=0}^{\infty} (\lambda - \mu)^n R_\lambda^{n+1},$$

   where A is the associated infinitesimal generator.

Section 2 (Extra practice questions, not for hand-in)

(a) Give an example of a Markov process which is not a martingale and of a martingale which is not a Markov process.

(b) Check that the transition functions of Brownian motion satisfy the Chapman-Kolmogorov equation.

(c) Prove that for Brownian motion on $\mathbb{R}$, and for $\lambda > 0$,

$$R_\lambda f(x) = \int_{\mathbb{R}} r_\lambda(x,y) f(y) dy, \qquad \text{where } r_\lambda(x,y) = \frac{1}{\gamma}\exp(-\gamma|y-x|), \text{ with } \gamma = \sqrt{2\lambda}.$$

(d) Suppose that X is a continuous time Markov process on a discrete state space (so can be characterised by a Q-matrix). Let $f_{ij}(t)$ denote the density of the first hitting time of state j if the chain starts in state i. Use the Markov property to find an integral equation which expresses the transition densities $p_{ij}(t)$ of the chain as a convolution of f_{ij} and p_{jj} and hence find an expression for the Laplace transform of the first hitting densities in terms of the resolvent of the chain.

(e) If X is a Feller process and f a non-negative function, check that

$$Y_t^\lambda = e^{-\lambda t} R_\lambda f(X_t), \qquad t \geq 0,$$

defines a supermartingale (with respect to distribution of X and the natural filtration), where R_λ is the resolvent corresponding to X.

(f) The *Cauchy process*, X, is the real-valued process for which $X_{s+t} - X_s$ is distributed as a Cauchy random variable with density

$$\frac{1}{\pi}\frac{t}{t^2 + x^2},$$

and increments corresponding to disjoint time intervals are independent.

Suppose that ϕ is an *odd* function, which is twice continuously differentiable with compact support and for which $\phi'(0) = 1$. Let $T(t)$ denote the expectation semigroup of X, that is $T(t)f(x) = \mathbb{E}[f(X_t)|X_0 = x]$. Suppose that f is twice continuously differentiable. Show that

$$\frac{T(t)f(x) - f(x)}{t} = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(x+y) - f(x) - f'(x)\phi(y)}{t^2 + y^2} dy,$$

and hence find an expression for the infinitesimal generator of $T(t)$.

Unlike the case of Brownian motion, the generator of the Cauchy process is not a local *operator. (An operator A is local if $Af(x)$ depends on the values of f only in an infinitesimal neighbourhood of x.) The Cauchy process does not have continuous paths, while Brownian motion does. In general, continuity of paths corresponds to locality of A.*