C8.2: Stochastic analysis and PDEs
Problem sheet 4

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The questions on this sheet are divided into two sections. Those in the first section are compulsory and should be handed in for marking. Those in the second are extra practice questions and should not *be handed in.*

Section 1 (Compulsory)

1. Let r satisfy the stochastic differential equation

$$dr_t = -\beta r_t dt + \sigma \sqrt{r_t} dW_t,$$

where $\{W_t\}_{t\geq 0}$ is standard $\mathbb{P}$-Brownian motion and $\beta, \sigma, r_0 > 0$.

Suppose that $\{u(t)\}_{t\geq 0}$ satisfies the ordinary differential equation

$$\frac{du}{dt}(t) = -\beta u(t) - \frac{\sigma^2}{2} u(t)^2, \quad u(0) = \theta,$$

for some constant $\theta > 0$. Fix $T > 0$. For $0 \leq t \leq T$ find the stochastic differential equation satisfied by

$$\exp\left(-u(T-t) r_t\right).$$

Hence find the moment generating function for r_T. Calculate the mean and variance of r_T and $\mathbb{P}[r_T = 0]$.

2. Use the Feynman-Kac stochastic representation formula to solve

$$\frac{\partial F}{\partial t}(t,x) + \frac{1}{2}\sigma^2 \frac{\partial^2 F}{\partial x^2}(t,x) = 0,$$

subject to the terminal value condition

$$F(T,x) = x^4.$$

3. We can use the Feynman-Kac representation to find the partial differential equation solved by the transition densities of solutions to stochastic differential equations.

Suppose that

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t. \tag{1}$$

For any set B let

$$p_B(t,x;T) \triangleq \mathbb{P}\left[X_T \in B \middle| X_t = x\right] = \mathbb{E}\left[\mathbf{1}_B(X_T) \middle| X_t = x\right].$$

Use the Feynman-Kac representation (assuming integrability conditions are satisfied) to write down an equation for

$$\frac{\partial p_B}{\partial t}(t, x; T).$$

By considering $B \to y$, deduce that the transition density $p(t, x; T, y)$ of the solution $\{X_s\}_{s\geq 0}$ to the stochastic differential equation (1) solves

$$\begin{aligned} \frac{\partial p}{\partial t}(t, x; T, y) + Ap(t, x; T, y) &= 0 \\ p(t, x; T, y) &\to \delta_y(x) \quad \text{as } t \to T, \end{aligned} \tag{2}$$

where A is the generator. Equation (2) is known as the *Kolmogorov backward equation* (it operates on the 'backward in time' variables (t, x)).

There is also a Kolmogorov forward equation acting on the *forward* variables (T, y). In the above notation,

$$\frac{\partial p}{\partial T}(t, x; T, y) = A^* p(t, x; T, y)$$

where A^* is the adjoint of A given by

$$A^* f(T, y) = -\frac{\partial}{\partial y}\left(\mu(T, y) f(T, y)\right) + \frac{1}{2}\frac{\partial^2}{\partial y^2}\left(\sigma^2(t, Y) f(T, y)\right).$$

4. Suppose that $\{X_t\}_{t\geq 0}$ solves

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t,$$

where $\{W_t\}_{t\geq 0}$ is a $\mathbb{P}$-Brownian motion. For $k : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ and $\Phi : \mathbb{R} \to \mathbb{R}$ given deterministic functions, find the partial differential equation satisfied by the function

$$F(t, x) \triangleq \mathbb{E}\left[\exp\left(-\int_t^T k(s, X_s)ds\right)\Phi(X_T)\middle| X_t = x\right],$$

for $0 \leq t \leq T$.

5. Let B be a Brownian motion in $\mathbb{R}$ and consider $A_t = \int_0^t I_{\{B_u > 0\}} du$, the amount of time that Brownian motion spends in the positive half line up to time t. Let $F(t, x) = \mathbb{E}(\exp(-\theta A_t)|B_0 = x)$, the Laplace transform of A_t given that the Brownian motion starts from x. By setting the dissipation term to be $r(t, B_t) = -\theta I_{\{B_t > 0\}}$ and the initial condition to be 1 and using a time reversed version of the Feynman-Kac formula, show the PDE satisfied by F, is

$$\frac{\partial F}{\partial t} = \begin{cases} \frac{1}{2}\frac{\partial^2 F}{\partial x^2} - \theta F & x > 0, t > 0 \\ \frac{1}{2}\frac{\partial^2 F}{\partial x^2} & x \leq 0, t > 0. \end{cases}$$

specifying the initial conditions and, carefully, the continuity conditions at 0. By taking Laplace transforms, $\hat{F}(\lambda, x) = \int_0^\infty \exp(-\lambda t)F(t, x)dt$ and solving the resulting ODE, show that

$$\hat{F}(\lambda, 0) = \frac{1}{\sqrt{\lambda}\sqrt{\lambda + \theta}}. \tag{3}$$

From this we can derive Levy's arcsine law,

$$P(A_t \leq s | X_0 = 0) = \int_0^s \frac{1}{\pi\sqrt{u(t-u)}} du = \frac{2}{\pi}\arcsin\left(\sqrt{\frac{s}{t}}\right), \quad 0 \leq s \leq t.$$

To see this compute the Laplace transform of the arcsine law by suitably integrating to show that the transform is as given in (3).

Section 2 (Extra practice questions, not for hand-in)

A. Consider a three dimensional Brownian motion started at the origin and stopped at the first time it exists the unit sphere. Fix $0 < r < 1$. In which of the annuli

$$A[a] = \{x \in \mathbb{R}^3 \,:\, a - r \leq |x| \leq a\} \text{ for } a \in [r, 1]$$

is the expected occupation time maximal?

B. Suppose that $v(t, x)$ solves

$$\frac{\partial v}{\partial t}(t,x) + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 v}{\partial x^2}(t,x) - rv(t,x) = 0, \quad 0 \leq t \leq T.$$

Show that for any constant θ,

$$v_\theta(t,x) \triangleq \frac{x}{\theta} v\left(t, \frac{\theta^2}{x}\right)$$

is another solution.

C. Suppose that for $0 \leq s \leq T$,

$$dX_s = \mu(s, X_s)ds + \sigma(s, X_s)dW_s, \quad X_t = x,$$

where $\{W_s\}_{t \leq s \leq T}$ is a $\mathbb{P}$-Brownian motion, and let $k, \Phi : \mathbb{R} \to \mathbb{R}$ be given deterministic functions. Find the partial differential equation satisfied by

$$F(t,x) = \mathbb{E}\left[\Phi(X_T) \middle| X_t = x\right] + \int_t^T \mathbb{E}\left[k(X_s) \middle| X_t = x\right] ds.$$

D. In the *Vasicek model*, the interest rate $\{r_t\}_{t \geq 0}$ is assumed to be a solution of the stochastic differential equation

$$dr_t = (b - ar_t)dt + \sigma dW_t,$$

where, as usual, $\{W_t\}_{t \geq 0}$ is standard $\mathbb{P}$-Brownian motion.

Find the Kolmogorov backward and forward differential equations satisfied by the probability density function of the process. What is the distribution of r_t as $t \to \infty$?

E. The process usually known as *Geometric Brownian motion* solves the SDE

$$dS_t = \mu S_t dt + \sigma S_t dW_t.$$

Find the forward and backward Kolmogorov equations for geometric Brownian motion and show that the transition density for the process is the lognormal density given by

$$p(t,x;T,y) = \frac{1}{\sigma y \sqrt{2\pi(T-t)}} \exp\left(-\frac{\left(\log(y/x) - \left(\mu - \frac{1}{2}\sigma^2\right)(T-t)\right)^2}{2\sigma^2(T-t)}\right).$$