QUESTION SHEET 1 – RANDOM MATRIX THEORY 2020/21

(1) Consider 2×2 real symmetric matrices

$$M = \begin{pmatrix} a & b \\ b & c \end{pmatrix}.$$

(a) Write down the GOE probability measure $P(M)\mathrm{d}M$ in terms of a, b and c.

(b) Let

$$O(\theta) = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}$$

be a **fixed** orthogonal matrix and set $M' = O^{\mathrm{T}} M O$. Show by explicit calculation that $P(M')\mathrm{d}M' = P(M)\mathrm{d}M$, i.e. that the GOE of 2×2 matrices is invariant under all orthogonal transformations.

(c) M can be written in the form $M = O_M^{\mathrm{T}} D O_M$, where

$$D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

with λ_1 and λ_2 being the eigenvalues of M, and where

$$O_M(\theta) = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}$$

is an orthogonal matrix that now depends on M.

(i) Write down formulae for the eigenvalues in terms of a, b and c and show that the probability that $\lambda_1 = \lambda_2$ is zero.

(ii) By changing variables from a, b, c to $\lambda_1, \lambda_2, \theta$, show that the probability measure can also be written in the form

$$P(M)\mathrm{d}M = C|\lambda_1 - \lambda_2| e^{-\frac{1}{4}(\lambda_1^2 + \lambda_2^2)} \mathrm{d}\lambda_1 \mathrm{d}\lambda_2 \mathrm{d}\theta$$

where C is a constant

(iii) Hence, show that the probability that $\lambda_1 - \lambda_2 \geq S$ can be written

$$\int_S^\infty p(s)\mathrm{d}s$$

where

$$p(s) = \frac{1}{4} s e^{-s^2/8}$$

(iv) Confirm this directly from the expressions for the probability measure and the eigenvalues in terms of a, b and c.

(v) Compute the mean value, $\bar{s}$, of $\lambda_1 - \lambda_2$ and so write down the probability density of the normalised spacing $x = s/\bar{s}$.

(2) Consider 2×2 complex Hermitian matrices

$$M = \begin{pmatrix} a & b + id \\ b - id & c \end{pmatrix}.$$

(a) Write down the GUE probability measure $P(M)\mathrm{d}M$ in terms of a, b, c and d.
(b) Derive an expression for the probability that $\lambda_1 - \lambda_2 \geq S$, where λ_1 and λ_2 are the two eigenvalues of M.
(c) Compute the mean value, $\bar{s}$, of $\lambda_1 - \lambda_2$, and so write down the probability density of the normalised spacing $x = s/\bar{s}$.
(d) Compare the result with the corresponding GOE expression when $x \to 0$.

(3) Consider again the 2×2 real symmetric matrix

$$M = \begin{pmatrix} a & b \\ b & c \end{pmatrix}.$$

By analyzing transformations $M' = O^{\mathrm{T}} M O$ where $O(\delta\theta)$ is close to the 2×2 identity matrix, show that the probability measures $\tilde{P}(M)\mathrm{d}M$ satisfying the requirements that (i) a, b and c are statistically independent random variables, and (ii) $\tilde{P}(M)\mathrm{d}M$ is invariant under orthogonal transformations, are those of the form

$$\tilde{P}(M) = \exp(-\kappa_1 \mathrm{Tr} M^2 + \kappa_2 \mathrm{Tr} M + \kappa_3)$$

where κ_1 is real and positive, and κ_2 and κ_3 are real. [Hint, expand to first order in $\delta\theta$.]

(4) Consider an undirected graph G with n vertices. Denote the adjacency matrix of G by A_n – i.e. A_n is an $n \times n$ symmetric matrix with entries $(A_n)_{ij} = 1$, if vertex i is adjacent to vertex j, and $(A_n)_{ij} = 0$, otherwise.

The Erdös-Rényi random graph $G(n, p)$ is formed by taking an empty graph on n vertices and adding each edge independently with probability p. Assume that $p \leq 1/2$ and that $np \to \infty$ when $n \to \infty$. Let J_n denote the $n \times n$ matrix all of whose entries are 1. Show that the eigenvalues of the matrix

$$M_n = \frac{1}{\sqrt{np(1-p)}}(A_n - pJ_n)$$

obey the semicircle law when $n \to \infty$. [NB, this also proves that the eigenvalues of $\frac{1}{\sqrt{np(1-p)}} A_n$ obey the semicircle law, because they interlace with those of M_n – see Lemma 39 of [1].]

(5) A d-regular graph is a graph where every vertex has degree d. Let $G_{n,d}$ be a d-regular graph chosen uniformly at random from all d-regular graphs on n labeled vertices. [Note that $G_{n,d}$ and $G\left(n, \frac{d}{n-1}\right)$ have the same edge density, but that the entries of the adjacency matrix of $G\left(n, \frac{d}{n-1}\right)$ are independent, while those of $G_{n,d}$ are not.]
(a) Let A_n denote the adjacency matrix of $G_{n,d}$. When d is fixed and $n \to \infty$ almost surely $G_{n,d}$ is locally a d-regular tree. Use this fact to show that, when $n \to \infty$, $\frac{1}{n}\mathrm{Tr}A_n^k$ is asymptotically equal to the number of closed walks of length k in an (infinite) d-regular tree starting at the root.
(b) Denoting

$$m_k = \lim_{n \to \infty} \frac{1}{n} \mathrm{Tr} A_n^k$$

show that $m_k = 0$ when k is odd, and that

$$m_{2k} = d \sum_{i=0}^{k-1} C_i (d-1)^i m_{2(k-1-i)}$$

where C_i is the ith Catalan number.

(c) Hence, using

$$\sum_{k=0}^{\infty} C_k x^k = \frac{1-\sqrt{1-4x}}{2x}$$

prove that

$$\sum_{k=0}^{\infty} m_{2k} y^k = \left(1 - \frac{d}{2(d-1)}(1-\sqrt{1-4(d-1)y})\right)^{-1}.$$

(d) Consider the probability density

$$f(x) = \begin{cases} \frac{d\sqrt{4(d-1)-x^2}}{2\pi(d^2-x^2)} & \text{if } |x| \leq 2\sqrt{d-1} \\ 0 & \text{if } |x| > 2\sqrt{d-1}. \end{cases}.$$

Denoting the moments of the associated measure by $\tilde{m}_k$, prove that $\tilde{m}_k = 0$ when k is odd, and that

$$\sum_{k=0}^{\infty} \tilde{m}_{2k} y^k = \left(1 - \frac{d}{2(d-1)}(1-\sqrt{1-4(d-1)y})\right)^{-1}.$$

(e) What conclusion do you draw about the empirical spectral density of random d-regular graphs in the limit when $n \to \infty$ with d fixed? [The formula you should write down is known as the Kesten-McKay law.]

(6) Make yourself familiar with MATLAB or MATHEMATICA, or any other programming language which allows you to generate random matrices and calculate eigenvalues[1] Do experiments to verify the semicircle law for random matrices from the GOE and the GUE, and for the adjacency matrices of Erdös-Rényi random graphs.

References

[1] T. Tao & V. Vu. Random matrices: universality of local eigenvalue statistics. *Acta Math.* **206**, 127?-204, 2011.

[1]This problem is optional and will not be marked. The same will be the case for similar problems on subsequent sheets that involve numerical computation. Nevertheless, you are strongly encouraged to try them.