QUESTION SHEET 2 – RANDOM MATRIX THEORY 2020/21

(1) Let X be an $n \times n$ complex Hermitian matrix taken at random from the GUE, scaled by $1/\sqrt{n}$ with respect to the definition in the lecture notes. Explicitly, $X_{ij} = X_{ji}^*$, for $1 \le i < j \le n$ the real and imaginary parts of X_{ij} are i.i.d. Gaussian random variables with mean 0 and variance $1/2n$, and X_{ii} are i.i.d. real Gaussian random variables with mean 0 and variance $1/n$ for $1 \le i \le n$.

(a) Prove that for any smooth function $f(\{X_{kl}\}_{k\le l})$, which does not grow too quickly as $|X_{kl}| \to \infty$,

$$\mathbb{E}\left[X_{ij} f(\{X_{kl}\}_{k\le l})\right] = \frac{1}{n}\mathbb{E}\left[\partial_{X_{ji}} f(\{X_{kl}\}_{k\le l})\right].$$

NB the partial derivative is computed with respect to the **complex** variable X_{ji}, defined so that if $z = x + iy$ and $g(z,\overline{z}) = g(x,y)$, then $\frac{\partial g}{\partial z} = \frac{1}{2}\left(\frac{\partial g}{\partial x} - i\frac{\partial g}{\partial y}\right)$ and $\frac{\partial g}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial g}{\partial x} + i\frac{\partial g}{\partial y}\right)$.

(b) Define $Y_k = \mathrm{Tr}X^k - \mathbb{E}\mathrm{Tr}X^k$. Prove, using the result of the previous question, for any integers $k_1, \dots, k_p$, the *Dyson-Schwinger equation* (also known as a *loop equation*)

$$\mathbb{E}\left[\mathrm{Tr}X^{k_1}\prod_{i=2}^{p} Y_{k_i}\right] = \mathbb{E}\left[\frac{1}{n}\sum_{l=0}^{k_1-2}\mathrm{Tr}X^l\mathrm{Tr}X^{k_1-2-l}\prod_{i=2}^{p} Y_{k_i}\right] + \mathbb{E}\left[\frac{1}{n}\sum_{i=2}^{p} k_i \mathrm{Tr}X^{k_1+k_i-2}\prod_{j=2,j\neq i}^{p} Y_{k_j}\right]$$

(c) Setting $m_k^{(n)} = \mathbb{E}\frac{1}{n}\mathrm{Tr}X^k$, prove that $|m_k^{(n)}| \le \kappa_k$, for finite constants κ_k, $k \in \mathbb{N}$.

(d) Prove that for all $k \in \mathbb{N}$

$$\mathbb{E}\left[\left(\frac{1}{n}\mathrm{Tr}X^k - \mathbb{E}\frac{1}{n}\mathrm{Tr}X^k\right)^2\right] = o(1)$$

when $n \to \infty$.

(e) Using the above results, prove that $m_k = \lim_{n\to\infty} m_k^{(n)}$ satisfies

$$m_k = \sum_{l=0}^{k-2} m_l m_{k-l-2}$$

with $m_0 = 1$ and $m_1 = 0$. Hence evaluate m_k.

(2) This problem covers the proof of the Marchenko-Pastur law using the Stieltjes transform. Let $X \in \mathbb{C}^{p\times n}$ be a random matrix with i.i.d. entries that we take to have zero mean, variance $1/n$, and eighth moment that is $O(1/n^4)$. Denote $R_p = XX^\dagger$. Let $y^\dagger \in \mathbb{C}^{1\times n}$ represent the first row of X and write

$$X = \begin{bmatrix} y^\dagger \\ Y \end{bmatrix}$$

(a) Show that for $z \in \mathbb{C}^+$

$$(R_p - zI_p)^{-1} = \begin{bmatrix} y^\dagger y - z & y^\dagger Y^\dagger \\ Yy & YY^\dagger - zI_{p-1} \end{bmatrix}^{-1}$$

(b) For $A \in \mathbb{C}^{p\times p}$ and $D \in \mathbb{C}^{n\times n}$, both invertible, and for $B \in \mathbb{C}^{p\times n}$ and $C \in \mathbb{C}^{n\times p}$,

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} (A - BD^{-1}C)^{-1} & -A^{-1}B(D - CA^{-1}B)^{-1} \\ -(A - BD^{-1}C)^{-1}CA^{-1} & (D - CA^{-1}B)^{-1} \end{pmatrix}.$$

Use this formula to show that

$$\left[(R_p - zI_p)^{-1}\right]_{11} = \frac{1}{-z - zy^\dagger(Y^\dagger Y - zI_n)^{-1}y}$$

(c) The following is a theorem that can be found in [1]. *Let $A_1, A_2, \ldots,$ with $A_N \in \mathbb{C}^{N\times N}$, be a series of matrices with uniformly bounded spectral norm. Let $x_1, x_2, \ldots,$ with $x_N \in \mathbb{C}^N$, be random vectors with i.i.d. entries of zero mean, variance $1/N$, and eighth order moment that is $O(1/N^4)$, independent of A_N. Then when $N \to \infty$, $x_N^\dagger A_N x_N - \frac{1}{N}\mathrm{Tr}A_N \xrightarrow{a.s.} 0$.*
Use this to show that

$$\left[(R_p - zI_p)^{-1}\right]_{11} - \frac{1}{-z - z\frac{1}{n}\mathrm{Tr}(Y^\dagger Y - zI_n)^{-1}} \xrightarrow{a.s.} 0.$$

(d) The following is a theorem that can be found in [1]. *Let $z \in \mathbb{C}\backslash\mathbb{R}$, $A \in \mathbb{C}^{N\times N}$, $B \in \mathbb{C}^{N\times N}$, with B Hermitian, and $v \in \mathbb{C}^N$. Then*

$$|\mathrm{Tr}((B - zI_N)^{-1} - (B + vv^\dagger - zI_N)^{-1})A| \le \frac{||A||}{|\mathrm{Im}z|}$$

where $||A||$ denotes the spectral norm of A. Moreover, if B is non-negative definite, for $z \in \mathbb{R}^-$

$$|\mathrm{Tr}((B - zI_N)^{-1} - (B + vv^\dagger - zI_N)^{-1})A| \le \frac{||A||}{|z|.}$$

Using this, show that, if $m(z)$ denotes the Stieltjes transform of the limiting spectral density of R_p when $p \to \infty$ and $p/n \to \gamma$, then m is a solution of

$$m = \frac{1}{1 - \gamma - z - z\gamma m}$$

and so confirm the Marchenko-Pastur law.

(3) Perform numerical experiments to test the Marchenko-Pastur law[1].

References

[1] Z. Bai & J.W. Silverstein, Spectral Analysis of Large Dimensional Random Matrices (Springer).

[1]This problem is optional and will not be marked. The same will be the case for similar problems on subsequent sheets that involve numerical computation. Nevertheless, you are strongly encouraged to try them.