QUESTION SHEET 3 – RANDOM MATRIX THEORY 2020/21

(1) This question sketches an alternative proof of Theorem 7 in the lecture notes (the joint eigenvalue probability density) for the GOE (when $\beta = 1$).

(a) Let M be an $n \times n$ real symmetric matrix with entries M_{ij}. One has that $M = O^{\mathrm{T}} D O$, where D is diagonal and O is an $n \times n$ orthogonal matrix. Verify that O can be written $O = e^{-H}$, where H is an $n \times n$ real *skew-symmetric* matrix (i.e. $H = -H^{\mathrm{T}}$, so $H_{ij} = -H_{ji}$ for all $1 \leq i, j \leq n$), by (i) confirming that e^{-H} is indeed an orthogonal matrix, and (ii) confirming that M and $O^{\mathrm{T}} D O$ share the same number of free parameters. You may assume that the eigenvalues of M are non-degenerate. [NB, $O = e^{-H}$ is a parameterization of the Lie group $\mathrm{SO}(N)$ by the Lie algebra $\mathrm{so}(N)$ of skew-symmetric matrices.]

(b) The goal is to compute the Jacobian J of the map $S : M \to O^{\mathrm{T}} D O$. Show that one can write

$$\mathrm{d}M = e^{H}[e^{-H}(\mathrm{d}e^{H})D + \mathrm{d}D - D(\mathrm{d}e^{-H})e^{H}]e^{-H}$$

This transports the calculation of the derivative at any arbitrary point e^{-H} to the identity element $I = e^{0}$ in the Lie group. The determinant of the Jacobian is preserved under this transformation and so it suffices to calculate it at $H = 0$ where $e^{H} = I$ and $\mathrm{d}e^{H} = \mathrm{d}H$.

(c) Confirm that at $H = 0$

$$\mathrm{d}M = (\mathrm{d}H)D + \mathrm{d}D - D(\mathrm{d}H).$$

(d) Hence compute $\det J$ and verify Theorem 7 in the notes for the GOE.

(2) This question sketches an alternative proof of Theorem 7 in the lecture notes (the joint eigenvalue probability density) for the GUE (when $\beta = 2$) along the same lines as in the previous problem. In this case M is complex Hermitian and one has that $M = U^{\dagger} D U$, where D is diagonal and U is an $n \times n$ unitary matrix. Show that U can be paramterized in the form $U = e^{-iH}$, where now H is itself an $n \times n$ *complex Hermitian* matrix. Hence verify Theorem 7 for the GUE.

(3) Let $p_k(x)$ be the kth Hermite polynomial. Confirm the Christoffel-Darboux identity

$$\sum_{k=0}^{n-1} \frac{p_k(x)p_k(y)}{k!} = \frac{p_n(x)p_{n-1}(y) - p_{n-1}(x)p_n(y)}{(n-1)!(x-y)}.$$

explicitly when $n = 1$ and $n = 2$, and hence prove the identity for general n by induction. [NB $p_0(x) = 1$, $p_1(x) = x$, and $p_2(x) = x^2 - 1$.]

(4) In the lectures we outlined a derivation of the semicircle law based on an asymptotic formula for the Hermite polynomials (see section 7.4). Fill in the steps omitted. You may wish to use the identity $\cos^2(\alpha + \phi) - \cos(\alpha)\cos(\alpha + 2\phi) = \sin^2(\phi)$.

(5) Let M be an $n\times n$ Hermitian matrix. Denote its characteristic polynomial $Q(\lambda)=\det(\lambda I - M)$. Prove that

$$\mathbb{E}\left[\prod_{j=1}^{k} Q(\lambda_j)\right] = \frac{1}{\Delta(\lambda_1,\dots,\lambda_k)}\det\begin{vmatrix} p_n(\lambda_1) & p_{n+1}(\lambda_1) & \dots & p_{n+k-1}(\lambda_1) \\ p_n(\lambda_2) & p_{n+1}(\lambda_2) & \dots & p_{n+k-1}(\lambda_2) \\ \vdots & \vdots & \ddots & \vdots \\ p_n(\lambda_k) & p_{n+1}(\lambda_k) & \dots & p_{n+k-1}(\lambda_k) \end{vmatrix}$$

where the expectation is computed with respect to the GUE, $\Delta(\lambda_1,\dots,\lambda_k)$ denotes the Vandermonde determinant, and $p_m(x)$ is the mth Hermite polynomial.

(6) Let $\mu(x)$ be a measure and let $\{f_k(x)\}_{k=1}^N$ and $\{g_k(x)\}_{k=1}^N$ be functions integrable with respect to $\mu(x)$. Prove the *Andréief identitiy*

$$\int\cdots\int \det\left[f_j(x_k)\right]\det\left[g_j(x_k)\right]\mathrm{d}\mu(x_1)\dots\mathrm{d}\mu(x_N) = N!\det\left[\int f_j(x)g_k(x)\mathrm{d}\mu(x)\right]$$

where all of the determinants are $N\times N$. Use this identity to prove the following.

(a) Denoting by $c_n^{(2)}$ the normalization constant in the joint eigenvalue probability density for the GUE (c.f. Theorem 7 in the lecture notes), show that

$$\frac{1}{c_n^{(2)}} = n!\det\left[2^{(j+k-3)/2}\left((-1)^{j+k}+1)\Gamma\left(\frac{1}{2}(j+k-1)\right)\right)\right]_{1\le j,k\le n}$$

(b) Confirm the normalization constant in the joint eigenvalue probability density for the CUE (Theorem 8 in the lecture notes) is correct.

(7) Perform numerical experiments to evaluate the two-point correlation function of matrices from the GUE and compare with the formula derived in the lectures. Evaluate as well the distribution of the largest eigenvalue and the spacing distribution between nearest-neighbouring eigenvalues[1] .

[1]This problem is optional and will not be marked. The same will be the case for similar problems on subsequent sheets that involve numerical computation. Nevertheless, you are strongly encouraged to try them.