QUESTION SHEET 4 – RANDOM MATRIX THEORY 2020/21

(1) Prove the (very useful!) transposing lemma for $n \times n$ determinants

$$\det[\phi_{j-1}(x_k)] \det[\psi_{j-1}(x_k)] = \det[\sum_{m=1}^{n} \phi_{m-1}(x_j)\psi_{m-1}(x_k)]$$

(2) Consider a function defined on the unit circle, $f(\phi) = \sum_{n\in\mathbb{Z}} f_n e^{in\phi}$, which is in $L^1(\mathrm{d}\phi)$. Let $\{e^{i\theta_k}\}_{k=1}^n$ denote the eigenvalues of an $n \times n$ random unitary matrix from the CUE. Prove the *Heine-Szegö* identity

$$\mathbb{E}\left[\prod_{k=1}^{n} f(\theta_k)\right] = D_{n-1}(f)$$

when the expectation is computed with respect to the CUE and where the *Toeplitz determinant* $D_{n-1}(f)$ is the determinant of the matrix

$$\begin{pmatrix} f_0 & f_1 & \cdots & f_{n-1} \\ f_{-1} & f_0 & \cdots & f_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ f_{-n+1} & f_{-n+2} & \cdots & f_0 \end{pmatrix}.$$

(3) Let A be an $n \times n$ unitary matrix. Prove that

(a) $\mathbb{E}\mathrm{Tr}A^k = n\delta_{k,0}$, where $\delta_{i,j}$ is the Kronecker delta symbol.

(b)

$$\mathbb{E}\left[|\mathrm{Tr}A^k|^2\right] = \begin{cases} n^2 & \text{if } k = 0 \\ |k| & \text{if } |k| \le n \\ n & \text{if } |k| > n \end{cases}$$

when the expectation is computed with respect to the CUE. [Hint: use the formula for the CUE 2-point correlation function derived in the lectures (section 7.7 in the notes).]

(c) $\mathbb{E}\left[\mathrm{Tr}A^k\overline{\mathrm{Tr}A^l}\right] = 0$ when $k \neq l$ and where, here and henceforth, $\overline{z}$ denotes the complex conjugate of z.

(4) (a) Consider a Poisson point process on $\mathbb{R}$. Let I be an interval of length $\mathcal{L}_I$ in which the expected number of points is $\mathcal{L}_I$ (so the process is such that the mean density of points is 1). Calculate the variance of the number of points in I.

(b) Let A be an $n \times n$ unitary matrix with eigenvalues $\{e^{i\theta_k}\}_{k=1}^n$. Prove that the expected number of eigenvalues in an interval I on the unit circle of length $\mathcal{L}_I$ is $\mathcal{L}_I n/2\pi$ and that the variance is $O(\log n)$. Compare this with the case of the Poisson point process. You may use without proof the identity

$$\sum_{j=-\infty}^{\infty}\sum_{k=1}^{n} \delta(\theta - \theta_k + 2\pi j) = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} \mathrm{Tr}A^m e^{im\theta}$$

where $\delta(\theta)$ denotes the Dirac delta function. You may also need to use the results of question (3).

(5) Let A be an $n \times n$ unitary matrix. Setting $Z(A,\theta) = \det(I - Ae^{-i\theta}) = \sum_{k=0}^{n} a_k e^{-ik\theta}$, i.e. $Z(A,\theta)$ is the characteristic polynomial of A, show that

(a)

$$\mathbb{E}\left[Z(A,\theta)\overline{Z(A,\phi)}\right] = \sum_{k=0}^{n} e^{ik(\phi-\theta)}$$

where the expectation is computed with respect to the CUE.

(b) and, hence, that $\mathbb{E}\left[a_n\overline{a_m}\right] = \delta_{nm}$

(6) Perform numerical experiments and plot your own pictures of Dyson Brownian Motion[1] .

(7) Compute numerically the 2-point correlation function of the zeros of the Riemann zeta-function, which you can find on Andrew Odlyzko's webpage and compare with the GUE/CUE formula.

(8) Compute numerically the distribution of the lengths of the longest increasing subsequences in randomly generated permutations of size n and verify the almost sure convergence to $2\sqrt{n}$.

[1]This problem is optional and will not be marked. The same will be the case for subsequent problems on this sheet that involve numerical computation. Nevertheless, you are strongly encouraged to try them.