Problem sheet 1 General Relativity II, Hilary Term 2021

Questions marked with a star have lowest priority to be discussed during class. Any comments or corrections please to Jan.Sbierski@maths.ox.ac.uk.

1) * **(Revision)** Let M be a smooth manifold and recall that $\mathfrak{X}^\infty(M)$ denotes the space of vector fields and $\Omega^1(M)$ the space of covector fields (1-forms). Show that a map

$$\tau : \underbrace{\mathfrak{X}^\infty(M) \times \cdots \times \mathfrak{X}^\infty(M)}_{\ell \text{ times}} \times \underbrace{\Omega^1(M) \times \cdots \times \Omega^1(M)}_{k \text{ times}} \to C^\infty(M)$$

is induced by a (k,ℓ)-tensor field if, and only if, it is multilinear over $C^\infty(M)$.

Similarly a map

$$\tau : \underbrace{\mathfrak{X}^\infty(M) \times \cdots \times \mathfrak{X}^\infty(M)}_{\ell \text{ times}} \times \underbrace{\Omega^1(M) \times \cdots \times \Omega^1(M)}_{k \text{ times}} \to \mathfrak{X}^\infty(M)$$

is induced by a $(k+1,\ell)$-tensor field if, and only if, it is multilinear over $C^\infty(M)$.

[*This is nearly trivial, just be careful with unravelling the definitions.*]

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2) Let $X, Y, Z \in \mathfrak{X}^\infty(M)$ be smooth vector fields on a Lorentzian manifold (M, g). Show that

$$(\nabla\nabla Z)(X,Y) - (\nabla\nabla Z)(Y,X) = \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]}Z$$

holds, where ∇ is the Levi-Civita connection. I.e., the left and the right hand side are equivalent definitions of the Riemann curvature tensor $R(X,Y)Z$.

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3) Let (M, g) be a Lorentzian (or Riemannian) manifold, consider a point $p \in M$ and let x^μ be a local coordinate system centred at p (i.e. $x^\mu(p) = 0$). Let $X, Y, Z \in T_pM$ be three tangent vectors. Let $0 < \varepsilon, \delta \ll 1$ be very small. We first parallely propagate Z along the curve γ, i.e., first from 0 along the straight coordinate line to εX^μ and then along the straight coordinate line to $\varepsilon X^\mu + \delta Y^\mu$ to obtain the vector $Z_\gamma(\varepsilon X^\mu + \delta Y^\mu)$. Now, we parallely propagate Z along the curve γ', i.e., first from 0 to δY^μ and then to $\varepsilon X^\mu + \delta Y^\mu$, both along straight coordinate lines. Denote the resulting vector by $Z_{\gamma'}(\varepsilon X^\mu + \delta Y^\mu)$. Show that to leading order in ε and δ we have

$$Z^\rho_\gamma(\varepsilon X^\mu + \delta Y^\mu) - Z^\rho_{\gamma'}(\varepsilon X^\mu + \delta Y^\mu) = -\varepsilon\delta R^\rho{}_{\kappa\alpha\beta}(0) Z^\kappa X^\alpha Y^\beta \ ,$$

thus giving another interpretation of curvature.

[*Hint: Can you justify that the parallel transport of Z from 0 to εX^μ is to leading order $Z^\mu - \varepsilon\Gamma^\mu_{\kappa\sigma}(0) X^\kappa Z^\sigma$?*]

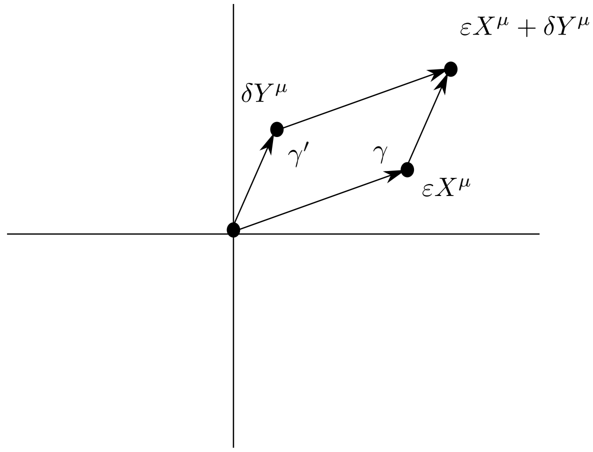

Figure 1: For problem 3

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4) * **(Revision)** Let M be a smooth manifold.

   (a) Let $X, Y, Z \in \mathfrak{X}^\infty(M)$. Prove the Jacobi identity

   $$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0 \ .$$

   (b) Let ∇ be an affine connection on M. Show that the torsion $T(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y]$ is a $(1,2)$-tensor field. Here $X, Y \in \mathfrak{X}^\infty(M)$.

   (c) Is the Lie bracket $[\cdot,\cdot] : \mathfrak{X}^\infty(M) \times \mathfrak{X}^\infty(M) \to \mathfrak{X}^\infty(M)$ an affine connection?

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5) Let M be a smooth manifold and let X, Y be smooth vector fields. Show that for a general (k,ℓ)-tensor field T we have the following identity for the Lie derivative:

   $$\mathcal{L}_X(\mathcal{L}_Y T) - \mathcal{L}_Y(\mathcal{L}_X T) = \mathcal{L}_{[X,Y]} T \ .$$

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6) (a) Let (M,g) be an n-dimensional Lorentzian (or Riemannian) manifold. Use the equation

   $$\nabla_a \nabla_b K_c = R^d{}_{abc} K_d \tag{1}$$

   from the lectures to show that the maximum number of linearly independent Killing vector fields on M is $\frac{n(n+1)}{2}$.

   [*Hint: Derive a system of ODEs with initial data given at a point p in M.*]

   (b) Consider 4-dimensional Minkowski spacetime. Write down equation (1) in standard Cartesian coordinates and derive the 10-dimensional space of Killing vectors on Minkowski spacetime. Choose the basis vectors such that they form the infinitesimal generators of translations and Lorentz transformations.

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7) This problem introduces and discusses *locally inertial coordinates*, which can be used by a freely falling observer to make contact with special relativity. We will make use of them later in the course when we discuss the observation of gravitational waves.

Let (M, g) be a $3+1$-dimensional Lorentzian manifold.

(a) Let y^μ be a coordinate system around a point $p \in M$. Show that

$$\partial_\kappa g_{\nu\rho}(p) = 0 \quad \text{for all } \nu, \rho, \kappa \in \{0, \ldots, 3\} \iff \Gamma^\mu_{\nu\kappa}(p) = 0 \quad \text{for all } \nu, \kappa, \mu \in \{0, \ldots, 3\} \, .$$

(b) Let $\gamma : I \to M$ be an affinely parameterised timelike geodesic with $g(\dot\gamma, \dot\gamma) = -1$, i.e., the worldline of a freely falling observer, parameterised by proper time. Consider a proper time $s_0 \in I$ and consider an orthonormal Lorentz frame $e_0 := \dot\gamma(s_0), e_1, e_2, e_3 \in T_{\gamma(s_0)}M$, i.e., we have $g(e_\mu, e_\nu) = m_{\mu\nu}$ with $m_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$. Parallely propagate the Lorentz frame e_μ along γ such that $\nabla_{\dot\gamma} e_\mu = 0$ holds. Introduce the mapping[1]

$$(x^0, x^1, x^2, x^3) \mapsto \exp_{\gamma(x^0)}(x^1 e_1 + x^2 e_2 + x^3 e_3) \in M \, .$$

Show that in a small neighbourhood of $\gamma(s_0)$ these are coordinates and that in these coordinates the metric takes the form

$$g = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2 + \mathcal{O}(r^2) dx^\mu \otimes dx^\nu \, ,$$

where $r^2 = (x^1)^2 + (x^2)^2 + (x^3)^2$. We call such coordinates *locally inertial coordinates.*

(c) Let $\gamma : I \to M$ be a timelike curve parametrised by proper time. Assume there exists a coordinate system x^μ in a neighbourhood of some point $\gamma(s_0)$ such that in these coordinates $\gamma(s) = (s, 0, 0, 0)$ and

$$g = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2 + \mathcal{O}(r^2) dx^\mu \otimes dx^\nu$$

holds, where r is as above. Show that, in particular, γ must be an affinely parametrised geodesic and that $\partial_1, \partial_2, \partial_3$ are parallel along γ.

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8) *(**Optional**) Let (M, g) be an n-dimensional *Riemannian* manifold.

(a) Let $p \in M$ and denote with $\Lambda^2 T^*_p M$ the space of all 2-covectors at p that are antisymmetric, i.e., all $\omega \in T^*_p M \otimes T^*_p M$ such that $\omega_{ab} = -\omega_{ba}$. Moreover, for $\alpha, \beta \in T^*_p M$ we define the wedge product $\alpha \wedge \beta := \alpha \otimes \beta - \beta \otimes \alpha \in \Lambda^2 T^*_p M$.

Let $\alpha^1, \ldots, \alpha^n$ be an orthonormal basis for $T^*_p M$. Show that $\alpha^i \wedge \alpha^j$ with $1 \leq i < j \leq n$ is a basis of $\Lambda^2 T^*_p M$ and thus $\Lambda^2 T^*_p M$ is $\frac{n(n-1)}{2}$ dimensional.

(b) Show that for $\alpha, \beta, \gamma, \delta \in T^*_p M$ the mapping

$$< \alpha \wedge \beta, \gamma \wedge \delta > := \det \begin{pmatrix} g^{-1}(\alpha, \gamma) & g^{-1}(\alpha, \delta) \\ g^{-1}(\beta, \gamma) & g^{-1}(\beta, \delta) \end{pmatrix}$$

induces an inner product on $\Lambda^2 T^*_p M$ with respect to which $\alpha^i \wedge \alpha^j$, $1 \leq i < j \leq n$, is an orthonormal basis. Also show that for $\omega, \rho \in \Lambda^2 T^*_p M$ one has $< \omega, \rho > = g^{ik} g^{jl} \omega_{ij} \rho_{kl}$.

[1]Recall from GR I that the exponential map $\exp_p : T_pM \supseteq U \to M$ at basepoint p maps the tangent vector X to $\exp_p(X) := \sigma(1)$, where $\sigma : [0, 1] \to M$ is the unique geodesic with $\sigma(0) = p$ and $\dot\sigma(0) = X$.

(c) Consider the Riemann curvature tensor as a $(2,2)$-tensor $R_{ij}{}^{kl}$ and show that it is a self-adjoint linear map $\mathbb{R} : \Lambda^2 T_p^* M \to \Lambda^2 T_p^* M$ with respect to the inner product $< \cdot, \cdot >$.

(d) Show that if (M, g) is connected and has $\frac{n(n+1)}{2}$ linearly independent Killing vector fields that then the Riemannian curvature tensor is of the form $R_{ijkl} = 2C g_{k[i} g_{j]l} = C(g_{ki} g_{jl} - g_{kj} g_{il})$ with C being a constant.

[Hint: You may use that an isometry $\phi : M \to M$ preserves the Riemann tensor, i.e., $(\phi^* R)_{ijkl} = R_{ijkl}$.]

Where in GR I have you encountered such spaces? One can indeed show further that manifolds whose Riemann tensor is of the above form with the same constant C are locally isometric.