Problem sheet 2 General Relativity II, Hilary Term 2021

Questions marked with a star have lowest priority to be discussed during class. Any comments or corrections please to Jan.Sbierski@maths.ox.ac.uk.

1) Let (M, g) be a *static* spacetime, i.e., there exists a timelike and hypersurface orthogonal Killing vector field V. Show that one can locally choose coordinates $\{x_0, x_1, \ldots, x_n\}$ such that

- $V = \frac{\partial}{\partial x_0}$
- $g_{\mu\nu}$ is independent of x_0
- $g_{0i} = 0$ for $i = 1, \ldots, n$.

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2) Consider the vector fields $V = \partial_x + y\partial_z$ and $W = \partial_y + x\partial_z$ in $\mathbb{R}^3$ with the standard Cartesian coordinates (x, y, z). Show that span$\{V, W\}$ is integrable and construct global coodinates (u, v, w) on $\mathbb{R}^3$ such that the integral manifolds are given as level sets of w.

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3) * **(Revision)** Let (M, g) be a Lorentzian manifold. Show that the Bianchi identity $\nabla_{[a} R_{bc]de} = 0$ implies the divergence freeness of the Einstein tensor, i.e.

$$\nabla^a (R_{ab} - \frac{1}{2} g_{ab} R) = 0 \, .$$

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4) Let (M, g) be a Lorentzian manifold and let T be a symmetric 2-covariant tensor field satisfying the conservation equation $\nabla^a T_{ab} = 0$. Let K be a Killing vector field on (M, g).

 (a) Show that the one-form $J(\cdot) = T(\cdot, K)$ is divergence free, i.e. $\nabla^a J_a = 0$.

 (b) Let now (M, g) be $3 + 1$ dimensional Minkowski spacetime with canonical $\{t, x, y, z\}$ coordinates and assume that T vanishes for $r = \sqrt{x^2 + y^2 + z^2}$ large enough. Show that for each Killing vector field K the corresponding charge

$$Q[K] := \int\limits_{t=t_0} J_0(t_0, x, y, z) \, dxdydz$$

 is conserved, i.e., independent of time.

 Express the conserved charges corresponding to the 10 linearly independent Killing vector fields found in problem 6*b*) from the first problem sheet in terms of the components of the stress-energy tensor. Give a physical interpretation of each of these charges.

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5) * Let (M, g) be a Lorentzian manifold.

 (a) Let $\phi \in C^\infty(M)$. We say that ϕ satisfies the wave equation iff $\Box_g \phi := \nabla^a \nabla_a \phi = 0$. Define the symmetric 2-covariant tensor field T associated to ϕ by

$$T(X, Y) := (X\phi)(Y\phi) - \frac{1}{2} g(X, Y) \cdot g^{-1}(d\phi, d\phi) \, .$$

Show that T is divergence free, i.e. $\nabla^a T_{ab} = 0$ if, and only if, ϕ satisfies the wave equation. Find the expression for T_{00} in terms of ϕ in the special case that (M, g) is the Minkowski spacetime.

(b) Let F be a two-form and define the associated symmetric 2-covariant tensor field

$$T_{ab} = \frac{1}{4\pi}(F_{ac}F_b{}^c - \frac{1}{4}g_{ab}F_{cd}F^{cd}) \ .$$

Show that T satisfies $\nabla^a T_{ab} = 0$ if F satisfies the Maxwell equations $dF = 0$ and $\nabla^a F_{ab} = 0$ (the other direction does in general not hold true). In the special case that (M, g) is the Minkowski spacetime find the expression for T_{00} in terms of the electric field $E_i = -F_{0i}$ and the magnetic field $B_i = \frac{1}{2}\varepsilon_{ijk}F_{jk}$.

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6) Let (M, g) be the $3+1$-dimensional Minkowski spacetime with Cartesian coordinates $\{x^0 = t, \underline{x}\}$, $\underline{x} = (x^1, x^2, x^3)$. Let T_{ab} be a symmetric 2-covariant tensor field which satisfies the conservation equation $\partial_a T^{ab} = 0$ and such that for every $t \in R$ we have that $T_{ab}(t, \cdot)$ is compactly supported in space (i.e. in $\mathbb{R}^3$). Define

$$P^i(t) := \int\limits_{\mathbb{R}^3} T^{0i}(t, \underline{x}) \ d\underline{x} \ , \qquad D^i(t) := \int\limits_{\mathbb{R}^3} T^{00}(t, \underline{x})x^i \ d\underline{x} \ , \qquad Q^{ij}(t) := \int\limits_{\mathbb{R}^3} T^{00}x^i x^j(t, \underline{x}) \ d\underline{x} \ ,$$

where $i, j = 1, 2, 3$. Show that the following holds:

$$\frac{d}{dt}D^i = P^i \ , \qquad \frac{d}{dt}P^i = 0 \ , \qquad \frac{d^2}{dt^2}Q^{ij} = 2\int\limits_{\mathbb{R}^3} T^{ij}(\cdot, \underline{x}) \ d\underline{x} \ .$$

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7) Let (M, g) be the $3+1$-dimensional Minkowski spacetime with Cartesian coordinates $\{x^0 = t, \underline{x}\}$, $\underline{x} = (x^1, x^2, x^3)$. Let T_{ab} be a symmetric 2-covariant tensor field which satisfies the conservation equation $\partial_a T^{ab} = 0$ and such that for every $t \in R$ we have that $T_{ab}(t, \cdot)$ is compactly supported in space (i.e. in $\mathbb{R}^3$). Moreover assume that the stress-energy distribution is stationary, i.e., that $\partial_t T_{ab} = 0$. Verify the following identities which are used in the lectures to derive the far field of an isolated gravitational body in the linear approximation.

(a) $(T^{lm}x^i x^j)_{,lm} = 2T^{ij}$ and show that this implies $\int\limits_{\mathbb{R}^3} T^{ij}(t, \underline{x}) \ d\underline{x} = 0$.

(b) Show that $\int\limits_{\mathbb{R}^3} T^{0j}(t, \underline{x}) \ d\underline{x} = 0$.

(c) $(T^i{}_j x^j)_{,i} = T^i{}_i$ and show that this implies $\int\limits_{\mathbb{R}^3} T^i{}_i(t, \underline{x}) \ d\underline{x} = 0$.

(d) $(T^{0i}x^j x^k)_{,i} = T^{0j}x^k + T^{0k}x^j$ and show that this implies $\int\limits_{\mathbb{R}^3} (T^{0j}x^k + T^{0k}x^j)(t, \underline{x}) \ d\underline{x} = 0$.

(e) $(T^k{}_i x^i x^j - \frac{1}{2}T^{jk}x_i x^i)_{,k} = T^i{}_i x^j$ and show that this implies $\int\limits_{\mathbb{R}^3} T^i{}_i(t, \underline{x})x^j \ d\underline{x} = 0$.

(f) $\frac{1}{|\underline{x}-\underline{x}'|_{\mathbb{R}^3}} = \frac{1}{r} + \frac{\underline{x}\cdot\underline{x}'}{r^3} + \mathcal{O}\big(\frac{1}{r^3}\big)$ as a function of $\underline{x}$, where $r^2 = \underline{x} \cdot \underline{x}$.

Note that the space indices $i, j, k, l, m, \ldots$ run from 1 to 3.

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8) Let $(\overline{M}, \overline{g})$ be a Riemannian manifold and let $M = \mathbb{R} \times \overline{M}$. Then $g = -dx_0^2 + \overline{g}$ is a Lorentzian metric on M. Show that $s \mapsto (\sigma^0(s), \overline{\sigma}(s))$ is an affinely parametrised geodesic in M if, and only if, $s \mapsto \overline{\sigma}(s)$ is an affinely parametrised geodesic in $\overline{M}$ and $\sigma^0(s) = \lambda \cdot s$ with $\lambda \in \mathbb{R}$.