Problem sheet 3 General Relativity II, Hilary Term 2021

Questions marked with a star have lowest priority to be discussed during class. Any comments or corrections please to Jan.Sbierski@maths.ox.ac.uk.

1) Consider the linearised Einstein equations in the wave gauge

$$\Box \bar{\tilde{h}}_{\mu\nu} = -16\pi \overset{(1)}{T}_{\mu\nu}$$
$$\partial^\mu \bar{\tilde{h}}_{\mu\nu} = 0 \; ,$$

where we recall that that $\bar{\tilde{h}}_{\mu\nu} = \tilde{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\tilde{h}$, where η is the Minkowski metric. Now consider a point mass (modelling a spherically symmetric body) of mass εM which is at rest at the coordinate origin $x = y = z = 0$. The corresponding stress energy tensor is given by $T_{\mu\nu} = \varepsilon \overset{(1)}{T}_{\mu\nu} = \varepsilon M \delta^3(\underline{x}) U_\mu U_\nu$, where $U = \partial_t$ is the four velocity of the particle. Derive the external gravitational field $g_{\mu\nu} = \eta_{\mu\nu} + \varepsilon \tilde{h}_{\mu\nu}$ under the assumption that it is stationary and compare it to the Schwarzschild metric with mass εM in the region where $\frac{\varepsilon M}{\sqrt{x^2+y^2+z^2}}$ is small.

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2) According to the Quadrupole formula the asymptotically leading order terms of the metric corrections $\tilde{h}_{\mu\nu}$ of a radiating system in wave gauge take the form

$$\bar{\tilde{h}}_{00}(t,\underline{x}) \simeq \frac{4M}{r} + \frac{2x_i x_k}{r^3}\frac{d}{dt^2}Q_{ik}(t-r)$$
$$\bar{\tilde{h}}_{0i}(t,\underline{x}) \simeq -\frac{2x_k}{r^2}\frac{d^2}{dt^2}Q_{ik}(t-r)$$
$$\bar{\tilde{h}}_{ij}(t,\underline{x}) \simeq \frac{2}{r}\frac{d^2}{dt^2}Q_{ij}(t-r) \; .$$

Recall that $\bar{\tilde{h}}_{\mu\nu} = \tilde{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\tilde{h}$, where η is the Minkowski metric. Compute the metric corrections $\tilde{h}_{\mu\nu}$ and show that the curvature component ${R^i}_{00j}$ of $g_{\mu\nu} = \eta_{\mu\nu} + \varepsilon\tilde{h}_{\mu\nu}$ to leading order in $0 < \varepsilon \ll 1$ and to leading order in $\frac{1}{r}$ is given by

$${R^i}_{00j}(t,\underline{x}) \simeq \frac{\varepsilon}{r}\Big[{\Pi^m}_i {\Pi^n}_j - \frac{1}{2}\Pi^{mn}\Pi_{ij}\Big]\frac{d^4}{dt^4}Q_{mn}(t-r) \; ,$$

where $\Pi^{mn} = \delta^{mn} - \frac{x_n}{r}\frac{x_m}{r}$. What is the interpretation of ${\Pi^m}_n$?

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3) Consider two stars, each of mass m and modelled as point particles, moving in a circular *Newtonian* orbit of radius R in the x,y-plane centred at the origin.

(a) Show that their trajectories may be taken to be $\gamma_\pm(t) = (t, \pm R\cos(\omega t), \pm R\sin(\omega t), 0)$, where $\omega^2 = \frac{m}{4R^3}$.

(b) Consider the corresponding stress-energy tensor

$$T^{\mu\nu} = \sum_{a=\pm} m \int \dot{\gamma}^\mu_a \dot{\gamma}^\nu_a \delta^4(x^\alpha - \gamma^\alpha_a(\tau))\, d\tau \; ,$$

where τ is proper time of the particles and $\dot{\gamma}^\mu_a = \frac{d}{d\tau}\gamma^\mu_a$. Compute the stress-energy tensor and the quadrupole moment in the slow motion approximation.

(c) Recall that our derivation of the quadrupole formula required the system to be non-self-gravitating, which is violated in this scenario. However, assume now that it still serves as a good approximation even in the weakly-self-gravitating case if all other assumptions made in its derivation are met. What restrictions does this impose on the parameters m and R?

Consider the stars to be each of one solar mass $\approx 2 \cdot 10^{30} kg$. Give an order of magnitude estimate on R for which we might expect the quadrupole formula to be a good approximation.

(d) $^{(*)}$ Compute the metric corrections $\bar{\bar{h}}_{\mu\nu}$ according to the quadrupole formula.

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4) This question recalls the derivation of electromagnetic radiation and thus makes explicit the differences and similarities between the gravitational and electromagnetic cases. Let (M, g) be the $3+1$-dimensional Minkowski spacetime with canonical coordinates x^μ. Maxwell's equation read $dF = 0$ and $\partial_\mu F^{\mu\nu} = 4\pi J^\nu$, where J is the source 4-vector and F is the Faraday tensor, an antisymmetric 2-form.

(a) Show that J satisfies the conservation law $\partial_\mu J^\mu = 0$ by virtue of the Maxwell equations.

(b) We now consider a source J that is compactly supported in space. Show that the charge $q(t) := \int_{\mathbb{R}^3} J^0(t, \underline{x})\, d\underline{x}$ is independent of time.

(c) Consider now the Maxwell equations in the Lorentz gauge

$$\begin{aligned} \Box \tilde{A}_\mu &= 4\pi J_\mu \\ \partial^\mu \tilde{A}_\mu &= 0 \end{aligned}$$

where $\tilde{A}$ is a one-form such that $d\tilde{A} = F$. Follow the approach taken in the gravitational case in the lectures to derive the following expression for the far field (i.e. for large $r = |\underline{x}|$)

$$\begin{aligned} \tilde{A}_0(t, \underline{x}) &\simeq \frac{q}{r} + \frac{x_i}{r^2} \frac{d}{dt} D_i(t - r) \\ \tilde{A}_i(t, \underline{x}) &\simeq -\frac{1}{r} \frac{d}{dt} D_i(t - r) \ , \end{aligned}$$

where $D_i(t) := \int_{\mathbb{R}^3} J^0(t, \underline{x}) x_i \, d\underline{x}$ is the *dipole moment*. Spell out the assumptions under which the above approximation is valid.

(d) $^{(*)}$ Compute the leading order of the electric and magnetic field in the far region and show that they (the leading orders) are orthogonal.

(e) $^{(*)}$ **(Distributional part is optional)** Consider a point particle with charge e oscillating according to

$$t \overset{\gamma}{\mapsto} (t, 0, 0, L \sin(\omega t)) \ .$$

The associated current 4-vector is given by $J^\mu = e \int \delta^4(x^\nu - \gamma^\nu(\tau)) \dot{\gamma}^\mu(\tau)\, d\tau$, where τ is the proper time of the particle and $\dot{\gamma}^\mu = \frac{d}{d\tau}\gamma^\mu$. Show that $\partial_\mu J^\mu = 0$ in the sense of distributions.

Assume now that the particle is moving slowly compared to the speed of light, i.e. $0 < \omega \ll 1$. Compute in this approximation the dipole moment $D_i(t)$.

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5) $^{(*)}$ Let (M, g) be a $(d+1)$-dimensional Lorentzian manifold and let $f : M \to \mathbb{R}$ be a function such that for $f^{-1}(0) =: \Sigma$ we have $df \neq 0$ on Σ, i.e., Σ is a hypersurface. Let now Σ be a *null* hypersurface, i.e., we have in addition $g^{-1}(df, df) = 0$ on Σ.

Show/recall that one can locally introduce coordinates $\{x_0, \dots, x_d\}$ such that $x_0 = f$. Let

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{01} & \cdots & g_{0d} \\ g_{10} & & & \\ \vdots & & g_{ij} & \\ g_{d0} & & & \end{pmatrix}$$

be the components of g in these coordinates. Show that $\det(\{g_{ij}\}_{1 \leq i,j \leq d}) = 0$.

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6) Let $M = \mathbb{R} \times (r_+, \infty) \times \mathbb{S}^2$ with the standard $\{t, r, \theta, \varphi\}$ coordinates where $r_+ = M + \sqrt{M^2 - a^2}$, $M > 0$, and $0 < a < M$. We define the Kerr metric g on M by

$$\begin{aligned} g = &-(1 - \frac{2Mr}{\rho^2})\, dt^2 - \frac{2Mra\sin^2\theta}{\rho^2}\,(dt \otimes d\varphi + d\varphi \otimes dt) + \frac{\rho^2}{\Delta}\, dr^2 \\ &+ \rho^2\, d\theta^2 + \left(r^2 + a^2 + \frac{2Mra^2\sin^2\theta}{\rho^2}\right)\sin^2\theta\, d\varphi^2 \,, \end{aligned}$$

where $\rho^2 = r^2 + a^2\cos^2\theta$ and $\Delta = r^2 - 2Mr + a^2$. Show that the vector field ∂_t is a Killing vector field and that it is timelike for $r > M + \sqrt{M^2 - a^2\cos^2\theta}$. Also show that ∂_t is not hypersurface orthogonal. Thus (M, g) is stationary but not static.