Finite Element Methods. QS 1

Lectures 1–5. Variational formulations, Sobolev spaces, Lax–Milgram.
Class: Week 4.

1. Let V be a real vector space. A bilinear form $a(u, v)$ is said to be *skew-symmetric* if $a(u, v) = -a(v, u)$. It is said to be *alternating* if $a(u, u) = 0$ for all $u \in V$.

(a) Show that every bilinear form on V may be written uniquely as the sum of a symmetric bilinear form and a skew-symmetric bilinear form.

(b) Show that a bilinear form on V is alternating if and only if it is skew-symmetric.

2. Let V be a real vector space and let (u, v) be an inner product on V. Let $\|\cdot\|$ be the induced norm. By considering the expansion of $\|u + v\|^2$, express the inner product purely in terms of norms.

3. Suppose $u : \Omega \to \mathbb{R}$ is a piecewise smooth function over a partition of a domain $\Omega = \bigcup_i K_i$, i.e. $u|_{\overline{K_i}} \in C^\infty(\overline{K_i})$. Show that $u \in C^0(\Omega) \implies u \in H^1(\Omega)$.

4. Given a vector field $u : \Omega \to \mathbb{R}^3$ on a domain $\Omega \subset \mathbb{R}^3$, its weak divergence $\nabla \cdot u : \Omega \to \mathbb{R}$ and curl $\nabla \times u : \Omega \to \mathbb{R}^3$ are defined by the following identities:

$$\int_\Omega u \cdot \nabla \phi \,\mathrm{d}x = -\int_\Omega \nabla \cdot u \phi \,\mathrm{d}x \quad \forall\, \phi \in C_0^\infty(\Omega),$$
$$\int_\Omega u \cdot \nabla \times \phi \,\mathrm{d}x = +\int_\Omega (\nabla \times u) \cdot \phi \,\mathrm{d}x \quad \forall\, \phi \in C_0^\infty(\Omega; \mathbb{R}^3).$$

(a) Prove that a piecewise smooth vector field u is in $H(\mathrm{div}, \Omega)$ if its normal component is continuous across cells.

(b) Prove that a piecewise smooth vector field u is in $H(\mathrm{curl}, \Omega)$ if its tangential components are continuous across cells.

5. Let $\Omega \subset \mathbb{R}^d$ $(d \in \{2, 3\})$ be an open bounded Lipschitz domain with boundary $\Gamma = \partial\Omega$ and outward-pointing boundary normal n. Let $f \in L^2(\Omega)$. Consider the sign-positive Helmholtz equation

$$\begin{aligned} -\nabla^2 u + k^2 u &= f \quad \text{in } \Omega, \\ \nabla u \cdot n &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

where $k \in \mathbb{R}$ with $|k| < 1$.

(a) State the variational formulation of this problem.

(b) For what values of k is the bilinear form coercive with respect to the norm $\|\cdot\|_{H^1(\Omega)}$? When the problem is coercive, give the coercivity constant of the bilinear form you have given in (a) in terms of k, and show that your coercivity bound is tight.

6. The proof of Lax–Milgram given in lectures is constructive: given a $u^0 \in V$, one can apply the fixed-point iteration

$$u^{k+1} = Tu^k, \quad k = 0, \ldots$$

with T defined in (4.2.18) of the notes. By the contraction mapping theorem, $u^k \to u$, the exact solution of the linear variational problem.

(a) For given coercivity and continuity constants α, C, what is the optimal choice of ρ in the construction of T?

(b) The constants for linear elasticity applied to steel are $\alpha = 75, C = 468$. How many iterations would be required to achieve a relative error of 0.1%?

(c) The constants for linear elasticity applied to rubber are $\alpha = 0.018, C = 2.75$. How many iterations would be required in this case for the same tolerance?

7. Suppose V is a Banach space, and that it has a coercive bounded bilinear form $a : V \times V \to \mathbb{R}$. Show that V is in fact a Hilbert space.

8. Let V be a Hilbert space. We say that a bilinear form $a : V \times V \to \mathbb{R}$ is *positive-definite* if $a(v, v) \geq 0$ for all $v \in V$.

Assume that a is positive-definite, bounded, symmetric, and that there exists a constant $\gamma > 0$ such that

$$\gamma \leq \inf_{\substack{u \in V \\ u \neq 0}} \sup_{\substack{v \in V \\ v \neq 0}} \frac{a(u, v)}{\|u\|_V \|v\|_V}.$$

Prove that these conditions imply that a is coercive. Characterise the coercivity constant of a in terms of its continuity constant and γ.