Finite Element Methods. QS 2

Lectures 1–11. Class: Week 6.

1. Let $\Omega = (0,1)^2$ with boundary $\partial\Omega$. Let $\Gamma_D = \{(x,y) \in \partial\Omega : x = 0\}$, and let $\Gamma_N = \partial\Omega \backslash \Gamma_D$. Consider the boundary value problem

$$\begin{aligned} -\nabla^2 u + 2u + \sin(x)u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \Gamma_D, \\ \nabla u \cdot n &= 0 \quad \text{on } \Gamma_N, \end{aligned}$$

where n denotes the outward unit normal to Γ_N.

Write this boundary value problem as:
(i) a variational problem over a function space V, to be defined;
(ii) the minimisation of an energy functional $J : V \to \mathbb{R}$;
(iii) an equation over elements of V^*, carefully introducing any necessary operators;
(iv) an equation over elements of V.

2. Consider the variational formulation: find $u \in H^1(\Omega)$ such that

$$\int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x + \alpha \int_{\partial\Omega} uv \, \mathrm{d}s = \int_\Omega vf \, \mathrm{d}x \quad \text{for all } v \in H^1(\Omega),$$

where $f \in L^2(\Omega)$ and real $\alpha > 0$.

Show that a solution u satisfying the Poisson equation $-\nabla^2 u = f$ in Ω and a Robin condition $\alpha u + \nabla u \cdot n = 0$ on the boundary $\partial\Omega$ satisfies the above weak formulation.

3. Consider the quadratic functional $J : v \in H^1(\Omega) \mapsto J(v) \in \mathbb{R}$ defined by

$$J(v) = \frac{1}{2} \int_\Omega \left(|\nabla v|^2 + v^2 \right) \mathrm{d}x + \frac{1}{2} \int_{\partial\Omega} v^2 \, \mathrm{d}s - \int_\Omega fv \, \mathrm{d}x - \int_{\partial\Omega} gv \, \mathrm{d}s.$$

State the associated optimality conditions, in variational and in strong form.

4. Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain. The equations of linear elasticity model the deformation of a structure with reference configuration Ω under an applied body force $f \in L^2(\Omega;\mathbb{R}^2)$. Specifically, the displacement $u \in H_0^1(\Omega;\mathbb{R}^2)$ (a vector field $u : \Omega \to \mathbb{R}^2$) is the minimiser of the energy functional

$$J(v) = \frac{1}{2}\int_\Omega 2\mu\varepsilon(v) : \varepsilon(v) + \lambda(\nabla \cdot v)^2 \,\mathrm{d}x - \int_\Omega f \cdot v \,\mathrm{d}x,$$

where $\lambda > 0$ and $\mu > 0$ are material-dependent constants, $\varepsilon(v) = \mathrm{sym}(\nabla v)$ with $\mathrm{sym}(A) = \frac{1}{2}\left(A + A^T\right)$, and for $A, B \in \mathbb{R}^{2\times 2}$, the Frobenius inner product is

$$A : B = A_{11}B_{11} + A_{12}B_{12} + A_{21}B_{21} + A_{22}B_{22}.$$

(a) Derive the linear variational problem that the minimiser $u \in H_0^1(\Omega;\mathbb{R}^2)$ must satisfy.

(b) Prove that if $A \in \mathbb{R}^{2\times 2}$ is symmetric, then $A : B = A : \mathrm{sym}(B)$. Use this to state the strong form of the optimality condition, assuming $u \in H^2(\Omega) \cap H_0^1(\Omega)$.

(c) Let the following norms and seminorms be given:

$$\|\varepsilon(v)\|_{L^2}^2 = \sum_{i,j=1}^{2} \|\varepsilon_{ij}(v)\|_{L^2}^2,$$
$$\|v\|_{H^1}^2 = \sum_{i=1}^{2} \|v_i\|_{H^1}^2,$$
$$|v|_{H^1}^2 = \sum_{i=1}^{2} |v_i|_{H^1}^2.$$

Prove that $\|\varepsilon(v)\|_{L^2}^2 \leq |v|_{H^1}^2$ and that $\|\nabla \cdot v\|_{L^2}^2 \leq 2|v|_{H^1}^2$, carefully justifying each step. Hence show that the bilinear form of the optimality condition of linear elasticity is continuous in the $\|\cdot\|_{H^1}$ norm.

[*Hint: note that* $(a-b)^2 \geq 0$ *for all* $a, b \in \mathbb{R}$.]

(d) You are told that the bilinear form of the optimality condition of linear elasticity is coercive in the $\|\cdot\|_{H^1}$ norm with coercivity constant μ. Consider the following table of material constants, where all values are in units of gigapascals:

material	μ	λ
steel	75	112
rubber	0.01	0.9
concrete	18	27

Based on your results, how do you expect the quality of a Galerkin approximation to vary for these different materials?

5. Consider a mesh $\mathcal{M}$ of $(0, 1)$ with 3 cells of equal size. Let $\mathcal{V}^{\mathrm{CG}_1}$, $\mathcal{V}^{\mathrm{DG}_1}$ and $\mathcal{V}^{\mathrm{CG}_2}$ be the function spaces constructed by equipping $\mathcal{M}$ with continuous linear Lagrange elements, discontinuous linear Lagrange elements and continuous quadratic Lagrange elements respectively. (Recall that *discontinuous* Lagrange elements are those where no continuity is enforced in the local-to-global map.) Sketch the global basis functions for these function spaces.

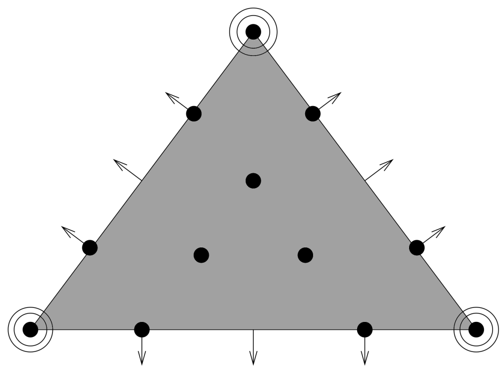
6. Consider the finite element given by $(K = [0, 1], \mathcal{P}_2(K), \mathcal{L})$, where $\mathcal{L} = (\ell_1, \ell_2, \ell_3)$ with

$$\begin{aligned} \ell_1(v) &= v'(0), \\ \ell_2(v) &= v'(1), \\ \ell_3(v) &= \int_0^1 v \, \mathrm{d}x. \end{aligned}$$

(a) Calculate the nodal basis functions for this element in terms of the monomial basis $\{1, x, x^2\}$.

(b) What is the maximal possible global continuity for function spaces constructed with this finite element? Justify your answer.

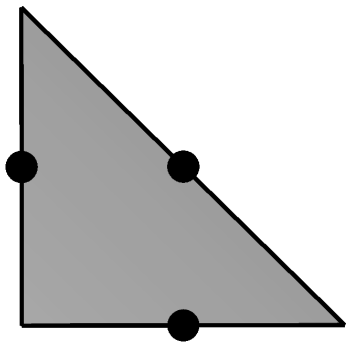
7. Consider the finite element $(K, \mathcal{V}, \mathcal{L})$ given by K a nondegenerate triangle, $\mathcal{V} = \mathcal{P}_7(K)$, and $\mathcal{L}$ shown in the figure below.



In this figure black dots refer to pointwise evaluation, a double ring refers to evaluation of the gradient and the Hessian, and the arrows along the edge refer to evaluation of the normal component of the gradient.

Prove that this element is unisolvent.

8. Consider the finite element $(K, \mathcal{V}, \mathcal{L})$, where K is a nondegenerate triangle with edges (e_1, e_2, e_3), $\mathcal{V} = \mathcal{P}_1(K)$, and $\mathcal{L} = (\ell_1, \ell_2, \ell_3)$, where $\ell_i(v)$ is the value of v at the midpoint of e_i. This element is depicted in the figure below and is referred to as the *Crouzeix–Raviart element of order* 1.



(a) Prove that this element is unisolvent.

(b) Let $\mathcal{M}$ be a triangular mesh of a polygonal domain $\Omega \subset \mathbb{R}^2$. Let $\mathcal{V}^{\mathrm{CR}_1}$ be the function space of maximal continuity constructed by equipping $\mathcal{M}$ with the Crouzeix–Raviart element of order 1. (Maximal continuity means that the degrees of freedom agree for triangles on either side of an edge.)

Let $\mathcal{V}^{\mathrm{CG}_1}$ and $\mathcal{V}^{\mathrm{DG}_1}$ be the function spaces constructed by equipping $\mathcal{M}$ with continuous and discontinuous linear Lagrange elements respectively. Establish that $\mathcal{V}^{\mathrm{CG}_1} \subsetneq \mathcal{V}^{\mathrm{CR}_1} \subsetneq \mathcal{V}^{\mathrm{DG}_1}$.

(c) Let K^+ and K^- be adjacent elements in $\mathcal{M}$, and let e be their common edge. Given $v \in \mathcal{V}^{\mathrm{CR}_1}$, denote by v^+ and v^- the restriction of v to K^+ and K^- respectively. Identify a function $f(v^+, v^-)$ such that

$$\int_e f(v^+, v^-) \,\mathrm{d}s = 0 \quad \text{for all } v \in \mathcal{V}^{\mathrm{CR}_1}.$$