Finite Element Methods. QS 3

Class: Week 8.

For some of these questions it may be more convenient to use a symbolic algebra package such as sympy, mathematica or maple.

1. Let $K = [0, 1]$, $\mathcal{V} = \mathcal{P}_3(K)$, and let $\mathcal{L} = \{\ell_0, \ell_1, \ell_0^*, \ell_1^*\}$, where

$$\begin{aligned}
\ell_0 &: v \mapsto v(0), \\
\ell_1 &: v \mapsto v(1), \\
\ell_0^* &: v \mapsto v'(0), \\
\ell_1^* &: v \mapsto v'(1).
\end{aligned}$$

This defines the first Hermite finite element in one dimension.

(a) Show that $\mathcal{L}$ determines $\mathcal{V}$.

(b) Construct the nodal basis $\{\phi_0, \phi_1, \phi_0^*, \phi_1^*\}$ for this finite element. Express your answers in the monomial basis.

(c) Consider the problem

$$-u'' + u = f, \quad u'(0) = 0 = u'(1).$$

Discretise $\Omega = [0, 1]$ into N intervals of uniform mesh size $h = 1/N$ and let V_h be the function space constructed by equipping each cell with the Hermite finite element defined above and imposing maximal global continuity. This induces a linear system

$$Ax = b.$$

(i) State formulae for the components of A and b.

(ii) For a given cell K, what is the local 4×4 matrix A_K of contributions to A?

(iii) Apply the finite element assembly algorithm cellwise to construct the matrix A for the case $N = 3$.

2. Consider the problem

$$-Tu'' = -\rho g, \quad u(0) = 0 = u(L),$$

with $L = 10$ m, $\rho = 1$ kg/m, $g = 9.8$ ms^{-2}, and $T = 98$ N. Here u is the vertical deflection of a hanging cable sagging under gravity. Compute the Galerkin approximation to this problem using the trial space

$$V_h = \text{span}\{\sin\left(\frac{\pi x}{L}\right), 1 - \cos\left(\frac{2\pi x}{L}\right), \sin\left(\frac{2\pi x}{L}\right)\}.$$

Note that each basis function of V_h satisfies the boundary conditions, and hence $u_h \in V_h$ will also.

3. A quadrature rule on a cell K has degree of precision m if it computes the exact answer for all polynomials of degree m or less.

What is the minimum degree of precision required to exactly assemble the matrix for the following bilinear forms? Furthermore, in one dimension, how many quadrature points n are required when using Gaussian quadrature?

(a) quadratic Lagrange elements, constant coordinate transformation Jacobian $J_K(\hat{x})$. Form:

$$a(u, v) = \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x.$$

(b) quintic Lagrange elements, linear coordinate transformation Jacobian $J_K(\hat{x})$. Form:

$$a(u, v) = \int_\Omega uv \, \mathrm{d}x.$$

4. Let K be a non-degenerate triangle, and let $\mathcal{V} = \mathcal{P}_2(K)$. Denote the vertices of K by z_i, $i = 1, \ldots, 3$. Let

$$\begin{aligned}
\ell_1 &: v \mapsto v(z_1), \\
\ell_2 &: v \mapsto v(z_2), \\
\ell_3 &: v \mapsto v(z_3), \\
\ell_4 &: v \mapsto v(\frac{z_1 + z_2}{2}), \\
\ell_5 &: v \mapsto v(\frac{z_1 + z_3}{2}), \\
\ell_6 &: v \mapsto v(\frac{z_2 + z_3}{2}).
\end{aligned}$$

Show that $\mathcal{L} = \{\ell_1, \ldots, \ell_6\}$ determines $\mathcal{V}$.

5. For $\Omega \subset \mathbb{R}^n$, $n \in \{1, 2, 3, 4\}$, consider the problem:

$$u = \underset{v \in H^1(\Omega)}{\operatorname{argmin}} \ J(v)$$

where

$$J(v) = \frac{1}{2} \int_\Omega \gamma \nabla v \cdot \nabla v + \frac{1}{2} \left(v^2 - 1\right)^2 \ \mathrm{d}x.$$

(i) State the Euler–Lagrange equation that characterises stationary points u of this functional.

(ii) Given an initial guess u_0, state the linearised system of equations that must be solved for the update δu in a Newton–Kantorovich iteration, in weak form.

6. Let V and Y be Banach spaces.

Consider the Newton–Kantorovich iteration applied to

(i) $F : V \to V^*$;

(ii) $(G \circ F) : V \to Y$, where $G : V^* \to Y$ is linear, continuous and invertible.

Prove that the Newton–Kantorovich iteration yields the same sequence of iterates in both cases when initialised from the same initial guess $u_0 \in V$.

Remark: this is the property of *affine covariance*, and is fundamental to the proper understanding of Newton-type methods. For a full discussion of this property, see P. Deuflhard, *Newton Methods for Nonlinear Problens*, Springer-Verlag, 2011.

7. Let $Z : V \to V$ be a continuous, invertible, linear operator. Suppose that it is a symmetry of a residual $F : V \to V^*$, i.e. it satisfies

$$ZRF(u) = RF(Zu)$$

for all $u \in V$, where $R : V^* \to V$ is the Riesz map. (Imagine, for example, that Z were a reflection; this property asserts that the reflection of the residual is the residual of the reflection.)

Prove that if the Newton–Kantorivich iteration is initialised from an initial guess u_0 that satisfies $Zu_0 = u_0$, then all subsequent iterates u_k will also satisfy $Zu_k = u_k$, so long as the iterates are defined.

Remark: this means that one should be careful when solving nonlinear problems with symmetries. In general, such problems will support both symmetric and nonsymmetric solutions. If the initial guess is chosen to be symmetric, then only symmetric solutions will be found by the iteration if the linear subproblem is solved exactly.