Finite Element Methods. QS 4

This sheet is not to be turned in. Complete it, and check your answers with the provided solutions.
Class: Trinity term.

1. In lectures we proved that for a stable discretisation of a stable (noncoercive) problem of the form

$$a(u, v) = F(v) \text{ for all } v \in V, \tag{T}$$

a Galerkin approximation satisfies the quasi-optimality result

$$\|u - u_h\|_V \leq (1 + c) \inf_{v_h \in V_h} \|u - v_h\|_V,$$

where u_h is the solution to the Galerkin approximation of (T) over a closed subspace $V_h \subsetneq V$. Here $c = C/\tilde{\gamma}$, where C is the continuity constant of a and $\tilde{\gamma}$ is the discrete inf-sup constant.

(i) Prove that (under the same conditions) the Galerkin approximation is stable, i.e. u_h satisfies

$$\|u_h\|_V \leq c\|u\|_V,$$

for the same constant $c = C/\tilde{\gamma}$.

(ii) For fixed $V_h \subsetneq V$ and a, consider the operator $P : V \to V_h$ defined by

$$a(u_h, v_h) = a(u, v_h) \quad \text{for all } v_h \in V_h.$$

In this equation we think of u as an input and $u_h = Pu$ as an output. Prove that P is linear and is a projection, i.e. $P^2 = P$.

(iii) A result from functional analysis states that for a bounded linear projection $P : V \to V$ satisfying $0 \neq P^2 = P \neq I$ (I the identity operator on V),

$$\|P\|_{\mathcal{L}(V,V)} = \|I - P\|_{\mathcal{L}(V,V)},$$

where the $\|\cdot\|_{\mathcal{L}(V,V)}$ norm is the operator norm

$$\|Q\|_{\mathcal{L}(V,V)} = \sup_{\substack{u \in V \\ u \neq 0}} \frac{\|Qu\|_V}{\|u\|_V}.$$

Using this result, derive the improved quasi-optimality estimate

$$\|u - u_h\|_V \leq c \inf_{v_h \in V_h} \|u - v_h\|_V.$$

2. Let $V = H_0^1(\Omega; \mathbb{R}^n)$ and $Q = L_0^2(\Omega)$. Let

$$L(u, p) = \frac{1}{2} \int_\Omega \nabla u : \nabla u \, \mathrm{d}x - \int_\Omega f \cdot u \, \mathrm{d}x - \int_\Omega p \nabla \cdot u \, \mathrm{d}x.$$

We say (u, p) is a saddle point of L iff

$$L(u, q) \leq L(u, p) \leq L(v, p)$$

for all $v \in V$, $q \in Q$.

Show that (u, p) is a weak solution of the Stokes equations if and only if it is a saddle point of the Lagrangian. (This is why these problems are called saddle point problems!)

3. Consider the mixed Poisson equation: find $(\sigma, u) \in H(\mathrm{div}, \Omega) \times L^2(\Omega)$ such that

$$\int_\Omega \sigma \cdot \tau \, \mathrm{d}x - \int_\Omega \nabla \cdot \tau u - \int_\Omega \nabla \cdot \sigma w \, \mathrm{d}x = - \int_\Omega f w \, \mathrm{d}x$$

for all $(\tau, w) \in H(\mathrm{div}, \Omega) \times L^2(\Omega)$.

(i) Write the mixed Poisson equation as the Fréchet derivative of a Lagrangian $L(\tau, w)$.

(ii) What constrained optimisation problem is encoded by this Lagrangian?

4. In this question we will investigate the key structure-preserving properties of the so-called *bounded cochain projections* π_V and π_Q used to prove the inf-sup inequality for the mixed Poisson equation in Lecture 15. We consider the complex

$$\begin{array}{ccc} H^1(\Omega; \mathbb{R}^2) & \xrightarrow{\mathrm{div}} & L^2(\Omega) \\ \Big\downarrow{\scriptstyle \pi_V} & & \Big\downarrow{\scriptstyle \pi_Q} \\ V_h & \xrightarrow{\mathrm{div}} & Q_h \end{array}$$

Here V_h is constructed on a triangular mesh with the Brezzi–Douglas–Marini element of degree 1: $K = \triangle$, $\mathcal{V} = \mathcal{P}_1(K)^2$, and degrees of freedom $\mathcal{L}$ defined by

$$\ell_{2i}(v) = \int_{e_i} v \cdot n \, \mathrm{d}s, \quad \ell_{2i+1}(v) = \int_{e_i} v \cdot nl \, \mathrm{d}s,$$

where e_i is the i^{th} edge of the triangle K, $i = 0, \ldots, 2$, n is the outward normal to the edge, and l is a fixed linear polynomial on the edge. (In other words, $\{1, l\}$ is a basis for $\mathcal{P}_1(e_i)$). Define π_V to be the finite element interpolation operator induced by this finite element. That is, the interpolant $\pi_V : H^1(\Omega; \mathbb{R}^2) \to V_h$ matches the zeroth and first order moments of the normal component of the interpolated function on each edge.

As in lectures, Q_h is constructed with the discontinous Lagrange element of degree 0: $K = \triangle$, $\mathcal{V} = \mathcal{P}_0(K) = \mathrm{span}(1)$, and

$$\mathcal{L} = \left\{ \ell : v \mapsto \int_\Omega v \, \mathrm{d}x \right\}.$$

Define π_Q to be the finite element interpolation operator induced by this finite element. In other words, $\pi_Q : L^2(\Omega) \to Q_h$ is the $L^2(\Omega)$-projection, given by

$$\int_\Omega (\pi_Q q) p_h \, \mathrm{d}x = \int_\Omega q p_h \, \mathrm{d}x \text{ for all } p_h \in Q_h.$$

(a) Show the commuting diagram property holds, i.e. that for any $\tau \in H^1(\Omega; \mathbb{R}^2)$,

$$\nabla \cdot (\pi_V \tau) = \pi_Q (\nabla \cdot \tau).$$

(b) We now turn to consider the boundedness of these cochain projections. Show that π_Q is bounded, i.e. for all $w \in L^2(\Omega)$,

$$\|\pi_Q w\|_{L^2(\Omega)} \leq \|w\|_{L^2(\Omega)}.$$

(c) Given the approximation results

$$\|\tau - \pi_V \tau\|_{L^2(\Omega)} \leq ch|\tau|_{H^1(\Omega)}, \quad \|w - \pi_Q w\|_{L^2(\Omega)} \leq ch\|w\|_{H^1(\Omega)},$$

show that π_V is bounded: there exists $c \in \mathbb{R}$ independent of h such that for all $\tau \in H^1(\Omega; \mathbb{R}^2)$,

$$\|\pi_V \tau\|_{H(\mathrm{div};\Omega)} \leq c\|\tau\|_{H^1(\Omega)}.$$

Here c is a generic constant that may take different values on different uses. [Hint: first bound $\|\pi_V \tau\|_{L^2(\Omega)}$ by writing $\pi_V \tau = \tau + \pi_V \tau - \tau$ and applying the triangle inequality.]

(d) Prove that if $\nabla \cdot \tau = 0$, then $\nabla \cdot \pi_V \tau = 0$ also.

5. Let $\Omega \subset \mathbb{R}^3$. It is desirable to construct a $H^2(\Omega)$-conforming finite element in three dimensions. Consider the following candidate:

Definition (*Tetrahedral Argyris element*). *Let K be a tetrahedron (4 vertices, 4 facets, 6 edges), let $\mathcal{V} = \mathcal{P}_5(K)$, and let the degrees of freedom $\mathcal{L}$ be defined as follows:*

- *Pointwise evaluation at 4 vertices.*
- *Pointwise evaluation at 4 interior points given in barycentric coordinates by $(\frac{5}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8})$, $(\frac{1}{8}, \frac{5}{8}, \frac{1}{8}, \frac{1}{8})$, $(\frac{1}{8}, \frac{1}{8}, \frac{5}{8}, \frac{1}{8})$ and $(\frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{5}{8})$.*
- *Derivative evaluation at 4 vertices.*
- *Hessian evaluation at 4 vertices.*
- *The derivative normal to the edge (two components), at the midpoint of 6 edges.*

(i) Show that this element is unisolvent.

(ii) Consider a facet F of the tetrahedron K with outward-pointing normal n. Do the degrees of freedom on F completely determine the normal derivative $\nabla u \cdot n$ on F? Is the tetrahedral Argyris element $H^2(\Omega)$-conforming?