Finite Element Methods. QS 4

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Class: Trinity term.

1. In lectures we proved that for a stable discretisation of a stable (noncoercive) problem of the form

$$a(u, v) = F(v) \text{ for all } v \in V, \tag{T}$$

a Galerkin approximation satisfies the quasi-optimality result

$$\|u - u_h\|_V \leq (1 + c) \inf_{v_h \in V_h} \|u - v_h\|_V,$$

where u_h is the solution to the Galerkin approximation of (T) over a closed subspace $V_h \subsetneq V$. Here $c = C/\tilde{\gamma}$, where C is the continuity constant of a and $\tilde{\gamma}$ is the discrete inf-sup constant.

(i) Prove that (under the same conditions) the Galerkin approximation is stable, i.e. u_h satisfies

$$\|u_h\|_V \leq c\|u\|_V,$$

for the same constant $c = C/\tilde{\gamma}$.

Solution: Applying the discrete inf-sup condition,

$$\begin{aligned}
\tilde{\gamma}\|u_h\|_V &\leq \sup_{\substack{v_h \in V_h \\ v_h \neq 0}} \frac{a(u_h, v_h)}{\|v_h\|_V} \\
&= \sup_{\substack{v_h \in V_h \\ v_h \neq 0}} \frac{a(u, v_h)}{\|v_h\|_V} \\
&\leq \sup_{\substack{v_h \in V_h \\ v_h \neq 0}} \frac{C\|u\|_V\|v_h\|_V}{\|v_h\|_V} \\
&= C\|u\|_V.
\end{aligned}$$

So $\|u_h\|_V \leq \frac{C}{\tilde{\gamma}}\|u\|_V$ as required.

(ii) For fixed $V_h \subsetneq V$ and a, consider the operator $P : V \to V_h$ defined by

$$a(u_h, v_h) = a(u, v_h) \quad \text{for all } v_h \in V_h.$$

In this equation we think of u as an input and $u_h = Pu$ as an output. Prove that P is linear and is a projection, i.e. $P^2 = P$.

Solution: Linearity: let $u, w \in V$. Let $F(v) := a(u, v)$ and $G(v) := a(w, v)$. Then $P(u + \lambda w)$ satisfies

$$a(P(u + \lambda w), v_h) = (F + \lambda G)(v_h) = F(v_h) + \lambda G(v_h) \quad \text{for all } v_h \in V_h,$$

for any $\lambda \in \mathbb{R}$, by definition. Similarly, Pu and Pw satisfy

$$a(Pu, v_h) = F(v_h), \quad a(Pw, v_h) = G(v_h).$$

Adding these two equations together, we have

$$a(Pu, v_h)+\lambda a(Pw, v_h) = a(Pu+\lambda Pw, v_h) = F(v_h)+\lambda G(v_h) = a(P(u+\lambda w), v_h).$$

Hence $P(u + \lambda w) = Pu + \lambda Pw$ as required.

Projection: again, let $u \in V$, $F(v) := a(u, v)$, $u_h := Pu$, and define $F_h(v) = a(u_h, v)$. By construction, $F_h(v_h) = F(v_h)$ for all $v_h \in V_h$. Then Pu_h satisfies

$$a(Pu_h, v_h) = F_h(v_h) = F(v_h) = a(Pu, v_h),$$

and hence $Pu_h = P^2u = Pu$.

(iii) A result from functional analysis states that for a bounded linear projection $P : V \to V$ satisfying $0 \neq P^2 = P \neq I$ (I the identity operator on V),

$$\|P\|_{\mathcal{L}(V,V)} = \|I - P\|_{\mathcal{L}(V,V)},$$

where the $\|\cdot\|_{\mathcal{L}(V,V)}$ norm is the operator norm

$$\|Q\|_{\mathcal{L}(V,V)} = \sup_{\substack{u\in V\\ u\neq 0}} \frac{\|Qu\|_V}{\|u\|_V}.$$

Using this result, derive the improved quasi-optimality estimate

$$\|u - u_h\|_V \leq c \inf_{v_h\in V_h} \|u - v_h\|_V.$$

Solution: Part (i) shows that $\|P\| \leq c$. Thus $\|I - P\| \leq c$ also. To derive the required result, we need to show that $\|u - u_h\|_V \leq \|I - P\|\|u - v_h\|_V$. Calculating, we find

$$u - u_h = u - Pu = u - Pu - v_h + Pv_h = u - v_h - P(u - v_h) = (I - P)(u - v_h)$$

for arbitrary $v_h \in V_h$. Thus (working forwards),

$$\begin{aligned}\|u - u_h\|_V &\leq \|I - P\|\|u - v_h\|_V \\ &= \|P\|\|u - v_h\|_V \\ &\leq c\|u - v_h\|_V,\end{aligned}$$

so that

$$\|u - u_h\|_V \leq c \inf_{v_h \in V_h} \|u - v_h\|_V.$$

as required.

2. Let $V = H_0^1(\Omega; \mathbb{R}^n)$ and $Q = L_0^2(\Omega)$. Let

$$L(u,p) = \frac{1}{2} \int_\Omega \nabla u : \nabla u \, \mathrm{d}x - \int_\Omega f \cdot u \, \mathrm{d}x - \int_\Omega p \nabla \cdot u \, \mathrm{d}x.$$

We say (u,p) is a saddle point of L iff

$$L(u,q) \leq L(u,p) \leq L(v,p)$$

for all $v \in V$, $q \in Q$.

Show that (u,p) is a weak solution of the Stokes equations if and only if it is a saddle point of the Lagrangian. (This is why these problems are called saddle point problems!)

Solution: Let

$$a(u,v) = \int_\Omega \nabla u : \nabla v \, \mathrm{d}x, \quad b(v,q) = -\int_\Omega q \nabla \cdot v \, \mathrm{d}x, \quad f(v) = \int_\Omega f \cdot v \, \mathrm{d}x.$$

We have that $L(u,p) = \frac{1}{2}a(u,u) + b(u,p) - f(u)$. Start with the first inequality:

$$\begin{aligned} \forall q \in Q,\ L(u,q) \leq L(u,p) &\Leftrightarrow \forall q \in Q,\ L(u,q) - L(u,p) \leq 0 \\ &\Leftrightarrow \forall q \in Q,\ b(u,q) - b(u,p) \leq 0 \\ &\Leftrightarrow \forall q \in Q,\ b(u,q-p) \leq 0 \\ &\Leftrightarrow \forall q \in Q,\ b(u,q) = 0, \end{aligned}$$

where in the last step we used the fact that Q is a vector space (take $q+p$ and $-q+p$).

Recall that if a is symmetric and coercive, then u solves $a(u,v) = f(v)$ for all $v \in V$ if and only if u minimises $J(v) = \frac{1}{2}a(v,v) - f(v)$ in V. Let $J_p(v) = \frac{1}{2}a(v,v) + b(v,p) - f(v)$. The second inequality can be rewritten:

$$\begin{aligned} \forall v \in V,\ L(u,p) \leq L(v,p) &\Leftrightarrow u \text{ minimises } J_p \text{ in } V \\ &\Leftrightarrow \forall v \in V,\ a(u,v) + b(v,p) = f(v), \end{aligned}$$

where the last line is the weak form of the momentum equation.

3. Consider the mixed Poisson equation: find $(\sigma, u) \in H(\mathrm{div}, \Omega) \times L^2(\Omega)$ such that

$$\int_\Omega \sigma \cdot \tau \, \mathrm{d}x - \int_\Omega \nabla \cdot \tau u - \int_\Omega \nabla \cdot \sigma w \, \mathrm{d}x = - \int_\Omega f w \, \mathrm{d}x$$

for all $(\tau, w) \in H(\mathrm{div}, \Omega) \times L^2(\Omega)$.

(i) Write the mixed Poisson equation as the Fréchet derivative of a Lagrangian $L(\tau, w)$.

Solution: By inspection, the required Lagrangian is

$$L(\tau, w) = \frac{1}{2} \int_\Omega |\tau|^2 \, \mathrm{d}x - \int_\Omega \nabla \cdot \tau w \, \mathrm{d}x + \int_\Omega f w \, \mathrm{d}x.$$

(ii) What constrained optimisation problem is encoded by this Lagrangian?

Solution: Recall that the minimisation of a quantity $J(\tau)$ subject to a constraint $C(\tau) = 0$ is related to the Lagrangian $L(\tau, w) := J(\tau) - (w, C(\tau))$. In this case, we have

$$J(\tau) = \int_\Omega |\tau|^2 \, \mathrm{d}x, \quad C(\tau) = \nabla \cdot \tau - f.$$

In other words, the optimisation problem is to compute

$$\sigma = \operatorname*{argmin}_{\tau \in H(\mathrm{div}, \Omega)} \frac{1}{2} \int_\Omega \tau \cdot \tau \, \mathrm{d}x,$$
$$\text{subject to } \nabla \cdot \tau = f.$$

In other words, we can think of the solution of the Poisson equation u as a Lagrange multiplier that enforces the constraint on the flux while minimising its $L^2(\Omega)$ norm.

4. In this question we will investigate the key structure-preserving properties of the so-called *bounded cochain projections* π_V and π_Q used to prove the inf-sup inequality for the mixed Poisson equation in Lecture 15. We consider the complex

$$\begin{tikzcd} H^1(\Omega; \mathbb{R}^2) \arrow[r, "\mathrm{div}"] \arrow[d, "\pi_V"] & L^2(\Omega) \arrow[d, "\pi_Q"] \\ V_h \arrow[r, "\mathrm{div}"] & Q_h \end{tikzcd}$$

Here V_h is constructed on a triangular mesh with the Brezzi–Douglas–Marini element of degree 1: $K = \triangle$, $\mathcal{V} = \mathcal{P}_1(K)^2$, and degrees of freedom $\mathcal{L}$ defined by

$$\ell_{2i}(v) = \int_{e_i} v \cdot n \, \mathrm{d}s, \quad \ell_{2i+1}(v) = \int_{e_i} v \cdot nl \, \mathrm{d}s,$$

where e_i is the i^{th} edge of the triangle K, $i = 0, \ldots, 2$, n is the outward normal to the edge, and l is a fixed linear polynomial on the edge. (In other words, $\{1, l\}$ is a basis for $\mathcal{P}_1(e_i)$). Define π_V to be the finite element interpolation operator induced by this finite element. That is, the interpolant $\pi_V : H^1(\Omega; \mathbb{R}^2) \to V_h$ matches the zeroth and first order moments of the normal component of the interpolated function on each edge.

As in lectures, Q_h is constructed with the discontinous Lagrange element of degree 0: $K = \triangle$, $\mathcal{V} = \mathcal{P}_0(K) = \text{span}(1)$, and

$$\mathcal{L} = \left\{ \ell : v \mapsto \int_\Omega v \, \mathrm{d}x \right\}.$$

Define π_Q to be the finite element interpolation operator induced by this finite element. In other words, $\pi_Q : L^2(\Omega) \to Q_h$ is the $L^2(\Omega)$-projection, given by

$$\int_\Omega (\pi_Q q) p_h \, \mathrm{d}x = \int_\Omega q p_h \, \mathrm{d}x \text{ for all } p_h \in Q_h.$$

(a) Show the commuting diagram property holds, i.e. that for any $\tau \in H^1(\Omega; \mathbb{R}^2)$,

$$\nabla \cdot (\pi_V \tau) = \pi_Q (\nabla \cdot \tau).$$

Solution: Both the left- and right-hand sides of this equation are piecewise constant functions. It therefore suffices to consider them over an arbitrary cell K of the mesh, i.e. we want to show that

$$\begin{aligned} \int_K \nabla \cdot (\pi_V \tau) \, \mathrm{d}x &= \int_K \pi_Q (\nabla \cdot \tau) \, \mathrm{d}x \\ &= \int_K \nabla \cdot \tau \, \mathrm{d}x, \end{aligned}$$

where the last equality holds because the indicator function for K is in Q_h.

Applying the divergence theorem, we have:

$$\begin{aligned} \int_K \nabla \cdot (\pi_V \tau) \, \mathrm{d}x &= \int_{\partial K} (\pi_V \tau) \cdot n \, \mathrm{d}s \\ &= \int_{\partial K} \tau \cdot n \, \mathrm{d}s \\ &= \int_K \nabla \cdot \tau \, \mathrm{d}x, \end{aligned}$$

where the middle equality holds by the definition of the BDM degrees of freedom.

(b) We now turn to consider the boundedness of these cochain projections. Show that π_Q is bounded, i.e. for all $w \in L^2(\Omega)$,

$$\|\pi_Q w\|_{L^2(\Omega)} \leq \|w\|_{L^2(\Omega)}.$$

Solution: We know that

$$\int_\Omega (\pi_Q w) p_h \, \mathrm{d}x = \int_\Omega w p_h \, \mathrm{d}x \text{ for all } p_h \in Q_h.$$

Choosing $p_h = \pi_Q w$, we have

$$\begin{aligned}\|\pi_Q w\|^2_{L^2(\Omega)} &= \int_\Omega w \pi_Q w \, \mathrm{d}x \\ &\leq \|w\|_{L^2(\Omega)} \|\pi_Q w\|_{L^2(\Omega)}.\end{aligned}$$

So $\|\pi_Q w\|_{L^2(\Omega)} \leq \|w\|_{L^2(\Omega)}$.

(c) Given the approximation results

$$\|\tau - \pi_V \tau\|_{L^2(\Omega)} \leq ch|\tau|_{H^1(\Omega)}, \quad \|w - \pi_Q w\|_{L^2(\Omega)} \leq ch\|w\|_{H^1(\Omega)} \text{ for } w \in H^1(\Omega),$$

show that π_V is bounded as a map from $H^1(\Omega, \mathbb{R}^2)$ to $V_h \subset H(\mathrm{div}; \Omega)$: there exists $c \in \mathbb{R}$ independent of h such that for all $\tau \in H^1(\Omega; \mathbb{R}^2)$,

$$\|\pi_V \tau\|_{H(\mathrm{div};\Omega)} \leq c\|\tau\|_{H^1(\Omega)}.$$

Here c is a generic constant that may take different values on different uses. [Hint: first bound $\|\pi_V \tau\|_{L^2(\Omega)}$ by writing $\pi_V \tau = \tau + \pi_V \tau - \tau$ and applying the triangle inequality.]

Solution: Applying the hint,

$$\begin{aligned}\|\pi_V \tau\|_{L^2(\Omega)} &\leq \|\tau\|_{L^2(\Omega)} + \|\tau - \pi_V \tau\|_{L^2(\Omega)} \\ &\leq \|\tau\|_{H^1(\Omega)} + ch|\tau|_{H^1(\Omega)} \\ &\leq \|\tau\|_{H^1(\Omega)} + c\,\mathrm{diam}(\Omega)|\tau|_{H^1(\Omega)} \\ &\leq (1 + c\,\mathrm{diam}(\Omega))\|\tau\|_{H^1(\Omega)}.\end{aligned}$$

Here we bounded $h \leq \mathrm{diam}(\Omega)$ to ensure that the boundedness constant is independent of the mesh size. (In other words, the continuity constant gets *better* as we refine the mesh, so we can take the worst possible h as our bound.)

Now, we consider

$$\begin{aligned}\|\pi_V \tau\|^2_{H(\mathrm{div};\Omega)} &= \|\pi_V \tau\|^2_{L^2(\Omega)} + \|\nabla \cdot \pi_V \tau\|^2_{L^2(\Omega)} \\ &= \|\pi_V \tau\|^2_{L^2(\Omega)} + \|\pi_Q \nabla \cdot \tau\|^2_{L^2(\Omega)} \\ &\leq c\|\tau\|^2_{H^1(\Omega)} + \|\nabla \cdot \tau\|^2_{L^2(\Omega)} \\ &\leq c\|\tau\|^2_{H^1(\Omega)}\end{aligned}$$

as required.

(d) Prove that if $\nabla \cdot \tau = 0$, then $\nabla \cdot \pi_V \tau = 0$ also.

Solution: Using the commuting diagram property,

$$\begin{aligned} \|\nabla \cdot (\tau - \pi_V \tau)\|_{L^2(\Omega)} &= \|\nabla \cdot \tau - \nabla \cdot \pi_V \tau\|_{L^2(\Omega)} \\ &= \|\nabla \cdot \tau - \pi_Q \nabla \cdot \tau\|_{L^2(\Omega)} \\ &\leq ch\|\nabla \cdot \tau\|_{H^1(\Omega)}. \end{aligned}$$

So if $\nabla \cdot \tau = 0$, then $\|\nabla \cdot (\tau - \pi_V \tau)\|_{L^2(\Omega)} = 0$, as required.

5. Let $\Omega \subset \mathbb{R}^3$. It is desirable to construct a $H^2(\Omega)$-conforming finite element in three dimensions. Consider the following candidate:

Definition (*Tetrahedral Argyris element*). *Let K be a tetrahedron (4 vertices, 4 facets, 6 edges), let $\mathcal{V} = \mathcal{P}_5(K)$, and let the degrees of freedom $\mathcal{L}$ be defined as follows:*

- *Pointwise evaluation at 4 vertices.*
- *Pointwise evaluation at 4 interior points given in barycentric coordinates by $(\frac{5}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8})$, $(\frac{1}{8}, \frac{5}{8}, \frac{1}{8}, \frac{1}{8})$, $(\frac{1}{8}, \frac{1}{8}, \frac{5}{8}, \frac{1}{8})$ and $(\frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{5}{8})$.*
- *Derivative evaluation at 4 vertices.*
- *Hessian evaluation at 4 vertices.*
- *The derivative normal to the edge (two components), at the midpoint of 6 edges.*

(i) Show that this element is unisolvent.

Solution: Suppose $u \in \mathcal{V}$ is such that all degrees of freedom evaluated on u are zero. On each facet, the degrees of freedom are the same as the triangular Argyris element, and hence by the unisolvence of the triangular Argyris element u must be zero on each of the facets. Hence u must factorise as

$$u = p\lambda_1\lambda_2\lambda_3\lambda_4,$$

where $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ are the barycentric coordinates on the tetrahedron, and p is a linear polynomial. However, u is zero at each of the four interior evaluation degrees of freedom, and hence p is also. Since p is a linear polynomial in three dimensions that is zero at four points, it must be zero, and hence $u = 0$ everywhere. Hence the element is unisolvent.

(ii) Consider a facet F of the tetrahedron K with outward-pointing normal n. Do the degrees of freedom on F completely determine the normal derivative $\nabla u \cdot n$ on F?

Is the tetrahedral Argyris element $H^2(\Omega)$-conforming?

Solution: The normal derivative $\nabla u \cdot n$ is a polynomial of degree 4 over K, and in particular on each facet of K. Let us examine whether the degrees of freedom on a given facet determine $\nabla u \cdot n$. Suppose all degrees of freedom evaluate to zero, and let us see if that forces $\nabla u \cdot n$ to be zero also. On each edge of the facet, the function is a quartic polynomial with a double root at the vertices and a single root at the midpoint, so it is zero on each edge. Thus we can write

$$\nabla u \cdot n = p\lambda_1\lambda_2\lambda_3,$$

where the edges are described by $\lambda_1 = 0$, $\lambda_2 = 0$, $\lambda_3 = 0$, and p is a polynomial of degree 1.

However, we have no data to constrain p: we have no extra degrees of freedom interior to the facet. In fact, $\nabla u \cdot n$ depends on the values of the 4 cell-interior degrees of freedom, which are not shared by adjacent cells. So $\nabla u \cdot n$ is *not* determined by the data shared between cells and the element is *not* $H^2(\Omega)$-conforming.