Proof of Second Order Necessary Optimality Conditions for Constrained Problems Theorem 19

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C6.2/B2: Continuous Optimization

Second-order optimality conditions

Theorem 19 (Second-order necessary conditions)
Let some CQ hold for (CP). Let x^* be a local minimizer of (CP), and (y^*, λ^*) Lagrange multipliers of the KKT conditions at x^*. Then

$$s^T \nabla^2_{xx} \mathcal{L}(x^*, y^*, \lambda^*) s \geq 0 \text{ for all } s \in F(\lambda^*),$$

where $\mathcal{L}(x, y, \lambda) = f(x) - y^T c_E(x) - \lambda^T c_I(x)$ is the Lagrangian function.

Proof of Theorem 19 (for equality constraints only) [**NON-EXAMINABLE**]:
Let $I = \emptyset$ and so $\mathcal{F}(x^*) = F(\lambda^*)$. We have to show that

$$s^T \nabla^2_{xx} \mathcal{L}(x^*, y^*, \lambda^*) s \geq 0 \text{ for all } s \text{ such that } J_E(x^*) s = 0.$$

As in the proof of Th16, we consider feasible perturbations/paths $x(\alpha)$ around x^*, where α (sufficiently small) scalar, $x(\alpha) \in \mathcal{C}^2(\mathbb{R}^n)$ and

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Proof of Theorem 19 (for equality constraints only) (continued)

$x(0) = x^*$, $x(\alpha) = x^* + \alpha s + \frac{1}{2}\alpha^2 p + \mathcal{O}(\alpha^3)$, and $c(x(\alpha)) = 0^{(\dagger)}$.

(†) requires constraint qualifications, namely, assuming the existence of $s \neq 0$, $p \neq 0$ with above properties.

For any $i \in E$, by Taylor's theorem for $c_i(x(\alpha))$ around x^*,

$$\begin{aligned} 0 &= c_i(x(\alpha)) = c_i(x^* + \alpha s + \tfrac{1}{2}\alpha^2 p + \mathcal{O}(\alpha^3)) \\ &= c_i(x^*) + \nabla c_i(x^*)^T(\alpha s + \tfrac{1}{2}\alpha^2 p) + \tfrac{1}{2}\alpha^2 s^T \nabla^2 c_i(x^*) s + \mathcal{O}(\alpha^3) \\ &= \alpha \nabla c_i(x^*)^T s + \tfrac{1}{2}\alpha^2 \left[\nabla c_i(x^*)^T p + s^T \nabla^2 c_i(x^*) s\right] + \mathcal{O}(\alpha^3). \end{aligned}$$

where we used $c_i(x^*) = 0$. Thus for all $i \in E$,

$$\nabla c_i(x^*)^T s = 0 \quad \text{and} \quad \nabla c_i(x^*)^T p + s^T \nabla^2 c_i(x^*) s = 0,$$

and so $J_E(x^*) s = 0$. Now expanding f, we deduce

$$\begin{aligned} f(x(\alpha)) &= f(x^*) + \nabla f(x^*)^T(\alpha s + \tfrac{1}{2}\alpha^2 p) + \tfrac{1}{2}\alpha^2 s^T \nabla^2 f(x^*) s + \mathcal{O}(\alpha^3) \\ &= f(x^*) + \alpha \nabla f(x^*)^T s + \tfrac{1}{2}\alpha^2 \left[\nabla f(x^*)^T p + s^T \nabla^2 f(x^*) s\right] + \mathcal{O}(\alpha^3). \end{aligned}$$

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Proof of Theorem 19 (for equality constraints only):(continued)

As x^* is a local minimizer, we must have that

$$\nabla f(x^*)^T p + s^T \nabla^2 f(x^*) s \geq 0. \quad (*)$$

From the KKT conditions, $\nabla f(x^*) = J_E(x^*)^T y^*$ and so

$$\nabla f(x^*)^T p = (y^*)^T J_E(x^*) p = -\sum_{i\in E} y_i^* s^T \nabla^2 c_i(x^*) s. \quad (**)$$

From (*) and (**), we deduce

$$\begin{aligned} 0 &\leq s^T \nabla^2 f(x^*) s - \sum_{i\in E} y_i^* s^T \nabla^2 c_i(x^*) s \\ &= s^T [\nabla^2 f(x^*) - \sum_{i\in E} \nabla^2 c_i(x^*)] s \\ &= s^T \nabla^2_{xx} \mathcal{L}(x^*, y^*) s. \qquad \square \end{aligned}$$