C6.2/B2. Continuous Optimization

Problem Sheet 4 – Solution of Problem 1

Problem 1. The fundamental theorem of linear inequalities, also known as *Farkas' Lemma* states that: given any vectors $b \in \mathbb{R}^n$ and $a_i \in \mathbb{R}^n$, $i \in \{1, \ldots, m\}$, the set

$$\{s : \; b^T s < 0 \quad \text{and} \quad a_i^T s \geq 0, i \in \{1, \ldots, m\}\}$$

is empty if and only if

$$b \in C = \{\sum_{i=1}^{m} a_i y_i : y_i \geq 0, \, i \in \{1, \ldots, m\}\}.$$

(In other words, a vector b lies in the cone C generated by the vectors a_i if and only if it cannot be separated from the vectors a_i by a separating hyperplane generated by s.) Use this lemma in the next part of the problem (for appropriate choices of b, a_i and m).

Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ and $c : \mathbb{R}^n \to \mathbb{R}^p$ are $\mathcal{C}^1$ functions. Let x^* be a local minimizer of

$$\min_x f(x) \quad c(x) \geq 0.$$

Show that, provided a suitable first-order constraint qualification holds, there exists a vector $\lambda_* \in \mathbb{R}^p$ of Lagrange multipliers such that

$$\nabla f(x^*) = J(x^*)^T \lambda^*, \quad c(x^*) \geq 0, \quad \lambda^* \geq 0, \quad \lambda_i^* c_i(x^*) = 0, \, i \in \{1, \ldots, p\}.$$

(These are the KKT conditions for inequality-constrained problems. Use ideas and approaches from the proof of Theorem 16; note that we only need a first-order representation of the feasible path in the proof of Theorem 16. Recall that it is sufficient to consider the active constraints at x^*.)

Solution. (*For your interest, a proof of Farkas' Lemma can be found in many textbooks; see for example, page 131 of (the recommended reading) NIM Gould, An Introduction to Algorithms for Continuous Optimization.*) As in the lectures, we are going to consider feasible paths/perturbations around x^*. We only need to consider active constraints at x^* (namely, $c_i(x^*) = 0$, $i \in \mathcal{A}$) since the inactive constraints (ie, $c_i(x^*) > 0$) remain inactive/strictly satisfied for sufficiently small perturbations around x^*. So we consider a vector-valued C^2 function

$$x(\alpha) = x^* + \alpha s + \mathcal{O}(\alpha^2) \text{ such that } x(0) = x^* \text{ and } c_i(x(\alpha)) \geq 0 \text{ for } \alpha \geq 0 \text{ suff. small and } i \in \mathcal{A}.$$

For $i \in \mathcal{A}$, we require that

$$\begin{aligned} 0 &\leq c_i(x(\alpha)) = c_i(x^* + \alpha s + \mathcal{O}(\alpha^2)) \\ &= c_i(x^*) + \alpha \nabla c_i(x^*)^T s + \mathcal{O}(\alpha^2) \\ &= \alpha \nabla c_i(x^*)^T s + \mathcal{O}(\alpha^2) \end{aligned}$$

where in the last equality we used $c_i(x^*) = 0$. Dividing the last displayed relation by $\alpha > 0$, we deduce

$$0 \leq \nabla c_i(x^*)^T s + \mathcal{O}(\alpha).$$

Letting $\alpha \longrightarrow 0$, we obtain

$$\nabla c_i(x^*)^T s \geq 0, \quad i \in \mathcal{A}, \tag{1}$$

which expresses the feasibility requirement on the directions s. Now expanding $f(x(\alpha))$ (same as in the proof of Theorem 19, except first-order expansion is sufficient), we obtain

$$f(x(\alpha)) = f(x^*) + \alpha \nabla f(x^*)^T s + \mathcal{O}(\alpha^2).$$

Along $x(\alpha)$, $f(\alpha)$ is essentially unconstrained and so x^* cannot be a local minimizer if (we have the same descent condition as in the unconstrained case, namely) $\nabla f(x^*)^T s < 0$. Thus, recalling (1), the set of feasible descent directions

$$\{s : \ \nabla f(x^*)^T s < 0 \qquad \nabla c_i(x^*)^T s \geq 0, \ i \in \mathcal{A}\}. \tag{2}$$

must be empty if x^* is a local minimizer. We can now apply Farkas' Lemma to the set in (2) with $b = \nabla f(x^*)$, $a_i = \nabla c_i(x^*)$ and $m = |\mathcal{A}|$. We deduce that there exists multipliers $\lambda_i \geq 0$, $i \in \mathcal{A}$, such that $\nabla f(x^*) = \sum_{i \in \mathcal{A}} \lambda_i \nabla c_i(x^*)$. Clearly, the complementarity conditions $\lambda_i c_i(x^*) = 0$ for all i hold for the already-defined multipliers corresponding to the active constraints $in \in \mathcal{A}$ since for those, $c_i(x^*) = 0$. For $i \notin \mathcal{A}$, let $\lambda_i = 0$. The feasibility requirement $c(x^*) \geq 0$ is clearly true for any minimizer of the constrained problem. **End of solution.**