# Problem Sheet 3: Suggested answers and hints

Derek Moulton

**TO BE HANDED IN BY FRIDAY OF WEEK 7: questions 6, 7, 8, 9 from Module II and questions 1 - 2 from Module III**

# Module II

## Question 6 - Fixed Volume/Pressure Constraint

We are asked to show that for a membrane under fixed pressure, $P$, or fixed volume, $V$, the shape equation, under the small gradient approximation, is given by:

$$\kappa \triangle\triangle h - \gamma \triangle h = P. \tag{1}$$

The idea is that the total energy of the membrane configuration does not only have a contribution from bending and stretching, as derived in the notes:

$$\varepsilon_2 = \frac{1}{2} \iint dxdy \left[ \kappa \left(\triangle h\right)^2 + \gamma \left(\nabla h\right)^2 \right], \tag{2}$$

but also a contribution from the work done by pressure, given by:

$$\varepsilon_P = -PV = -P \int_\Omega dV, \tag{3}$$

. where the pressure is assumed to be constant and $\Omega$ is the integral over the volume of the membrane. In the case of a fixed volume $V = V_0$, we would add the following contribution to the energy

$$-P(\int_\Omega dV - V_0), \tag{4}$$

where the constant $P$ (still a pressure!) is playing the role of a Lagrange multiplier enforcing the fixed volume. Since $V_0$ is constant both scenarios are identical in the analysis below.

Using the hint from the problem sheet, we introduce the divergence of the position vector, $\boldsymbol{r} = (x, y, h(x, y))$, into equation (3):

$$\varepsilon_P = -P \int_\Omega dV = -\frac{P}{3} \int_\Omega \nabla \cdot \boldsymbol{r} dV = -\frac{P}{3} \int_\Sigma \boldsymbol{r} \cdot \boldsymbol{n} dS, \tag{5}$$

where we have used the divergence theorem and are now finding the integral over the area of the membrane, $\Sigma$.

The unit normal vector is given by:

$$\boldsymbol{n} = \frac{1}{\sqrt{1 + \left(\nabla h\right)^2}} \left(-h_x, -h_y, 1\right), \tag{6}$$

so evaluating in the Monge parametrization, we find:

$$\varepsilon_P = \cdots = \frac{P}{3} \iint dxdy \left\{ \boldsymbol{r}_2 \cdot \nabla h - h \right\}, \tag{7}$$

where $\boldsymbol{r}_2 = (x, y)$.

The problem now becomes a matter of minimizing the functional:

$$\varepsilon_{Total} = \varepsilon_2 + \varepsilon_P = \frac{1}{2} \iint dxdy \left[ \kappa \left( \triangle h \right)^2 + \gamma \left( \nabla h \right)^2 \right] + \frac{P}{3} \iint dxdy \left\{ \boldsymbol{r}_2 \cdot \nabla h - h \right\}. \tag{8}$$

Consider: what constraints or boundary conditions are needed?

## Question 7 - Shape Equation over a Step Function

We are asked to find the shape of a membrane that is hanging over a step-edge. To do this, we work in $1D$ using the corresponding shape equation derived in lectures:

$$\frac{d^4h}{dx^4} - \frac{1}{\lambda^2}\frac{d^2h}{dx^2} = 0, \tag{9}$$

Now think about the correct boundary conditions, and then solving is straightforward...

There are two length scales in the problem: $L$ and $\lambda$. Plot the solution for various values of $L$ and $\lambda$. What do you notice about the solution when $L \gg \lambda$? What about when $\lambda \gg L$? What sort of energy is minimized in both scenarios?

## Question 8 - Pressure and Surface Tension of Cylindrical Vesicles

To begin with, let us consider the generalized energy functional for a membrane under constant pressure:

$$\varepsilon = \int_{\Sigma} dS \left[\gamma + 2\kappa \left(H - H_0\right)^2 + \bar{\kappa} K_G\right] - P \int_{\Omega} dV, \tag{10}$$

where $\Sigma$ is the surface of our membrane and $\Omega$ is its associated volume.

When dealing with a cylindrical surface, we find that the mean and Gaussian curvatures are $H = \frac{1}{2R}$ and $K_G = 0$ respectively, where $R$ is our radial coordinate. Where do these results come from? What are the principal curvatures of a cylinder?

Taking this and the generalization that $\kappa = 1$, the energy functional becomes

$$\varepsilon = \int_{\Sigma} dS \left[\gamma + 2\left(\frac{1}{2R} - H_0\right)^2\right] - P \int_{\Omega} dV. \tag{11}$$

Now, using the parametrization in cylindrical coordinates $(R, \theta, z)$, you can exactly evaluate these integrals...

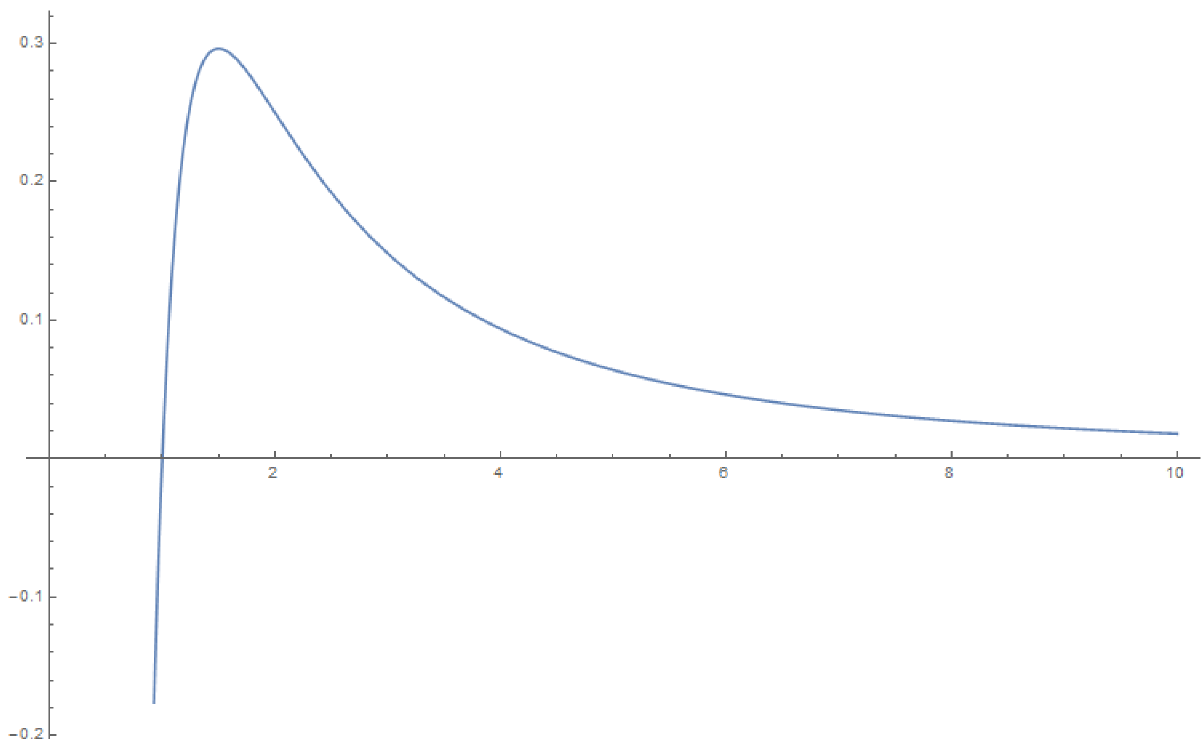
*For consideration: There is something that we are neglecting in our energy functional. What part of the cylindrical vesicle aren't we including? What assumption about our vesicle can we make for the above to hold?*

Now, minimise the energy as a function of radius and length...

You should find that the energy minimisation is independent of the length $L$. *Why?*

Finally, you can solve for $P$ and $\gamma$ giving equations of the form $P = P(R)$, $\gamma = \gamma(R)$.

As a sample, plotting $P$ for $H_0 = 0.5$ over $R \in [0, 10]$ yields the following:



In general, we find that $P(R)$ has a maximum which occurs when $R = \frac{3}{4H_0}$.

*Is the possibility of an instability evident by this calculation and/or in the figure? Suppose that we can control the pressure of this membrane, but we can't control its radius (i.e. like blowing up a cylindrical balloon). What happens as we continue to increase the pressure?*

## Question 9 - The Sea Urchin

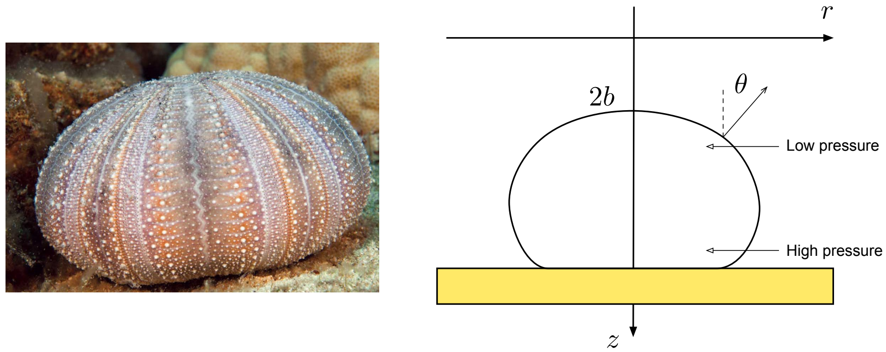



Figure 1: The shape of a sea urchin is a balance between a pressure gradient and tension.

The assumptions in the problem are that pressure $P = \rho g z$ and that the tensions $t_s$ and $t_\phi$ are equal. Consider what this implies in the force balance equations:

$$\frac{d}{ds}(r t_s) = r'(s) t_\phi, \tag{12}$$

$$P = t_s \kappa_s + t_\phi \kappa_\phi \tag{13}$$

The boundary conditions are $\theta(0) = 0,\ r(0) = 0,\ z(0) = 2b$, but due to the singularity in $\theta'$ we we need to expand this to define at $s = \epsilon$. To do this, we use a 'backwards' Taylor expansion, e.g.

$$r(0) = r(\epsilon) - r'(\epsilon)\epsilon + O(\epsilon^2), \ldots$$

Solutions for two different values of $a$ are shown in the figure below. We see that for small $a$, the shape is very squamous, while for large $a$ it is nearly spherical, though much smaller in size.

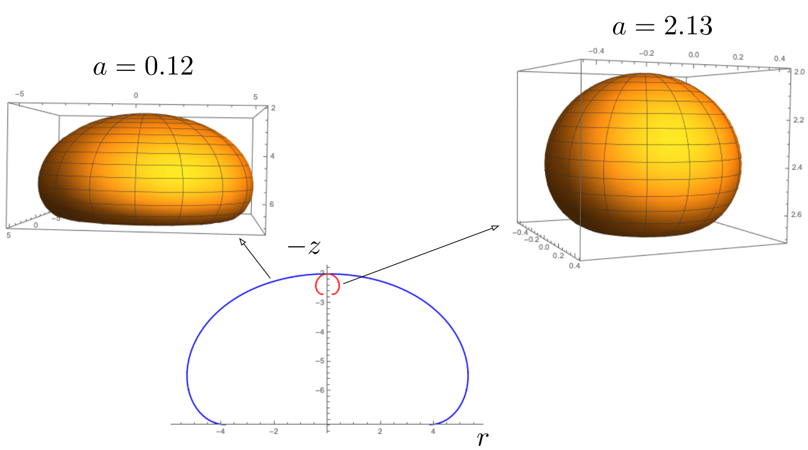



Figure 2:

# Module III

## Question 1 - 1D Growing Rod in Gravity

We consider a vertical rod of constant density $\rho$ in a gravitational field with strength $g$, and which follows the growth law:

$$\frac{\partial \gamma}{\partial t} = \kappa\gamma\left(\sigma - \sigma^{\star}\right). \tag{14}$$

We first work out the stress that is created by the gravitational body force:

$$\frac{\partial \sigma}{\partial S} = -\rho g \implies \sigma = -\rho g S, \tag{15}$$

where we have used the boundary condition that no stress acts on the top-most end of our rod (i.e. at $S = 0$). Why is the above differential equation with respect to $S$ and not $s$ or $S_0$?

Next, since $\gamma$ is the growth stretch of our rod, we can relate the reference configuration and the unstressed growth configuration by:

$$\gamma = \frac{\partial S}{\partial S_0} \implies S = \int_0^{S_0} \gamma dS_0. \tag{16}$$

Now differentiate this with respect to time and substituting the growth law and the form of $\sigma$, and convert to the grown configuration $dS$. Integrating the result gives:

$$\frac{\partial S}{\partial t} = -\kappa\left[\frac{\rho g S^2}{2} + \sigma^{\star} S\right], \tag{17}$$

which is a separable differential equation that can be solved and rearranged for $S = S(S_0, t)$.

Now, turn to the constitutive relation, $\sigma = E\left(\alpha - 1\right)$, noting that $\alpha = \partial s/\partial S$, and use this to obtain an expression for $s$ in terms of $S(S_0, t)$. Thus we have an expression for $s(S_0, t)$ and the total length satisfies $l(t) = s(L_0, t)$.

Taking $t \to \infty$ (and noting $\sigma^{*} < 0$) then yields the required result.

Is this a realistic model?

## Question 2 - The Revenge of the 1D Growing Rod in Gravity

Here we consider a new growth law:

$$\frac{\partial \gamma}{\partial t} = \kappa \gamma \left(\sigma - \sigma^{\star}\right) H\left(S - (L - R)\right) H\left(\sigma - \sigma^{\star}\right), \tag{18}$$

which confines growth to a region, $R$, at the bottom of the rod and also imposes that growth only acts when the stress exceeds the homeostatic stress (i.e. when the rod is in tension relative to the homeostatic compressive stress $\sigma^{*} < 0$). Furthermore, the homeostatic stress and the region of size $R$ satisfy the inequality $L - R < \frac{-\sigma^{\star}}{\rho g}$, where $L$ is the reference length of the growing rod.

Following similar steps as in the previous problem, we get

$$\frac{\partial L}{\partial t} = -\kappa \int_{0}^{L} \left(\rho g S - \rho g S^{\star}\right) H\left(S - (L - R)\right) H\left(\sigma - \sigma^{\star}\right) dS, \tag{19}$$

where $S^{\star}$ is the $S$ coordinate associated with $\sigma^{\star}$.

Now consider the Heaviside functions, keeping in mind that the stress $\sigma$ is a monotonically decreasing function of $S$. The integral simplifies to

$$\frac{\partial L}{\partial t} = -\kappa \rho g \int_{L-R}^{S^{\star}} \left(S - S^{\star}\right) dS. \tag{20}$$

(Can you see why?)

Now integrate, separate variables, and integrate again to get

$$-\frac{1}{\left(L - R - S^{\star}\right)} + \frac{1}{\left(L_0 - R - S^{\star}\right)} = \frac{\kappa \rho g}{2} t, \tag{21}$$

which can be solved for $L(t)$.

Lastly, we can obtain an expression for the current length $l(t)$ (following both growth and loading) by integrating

$$\alpha = \frac{ds}{dS}$$

from $S = 0$ to $L(t)$ and using the constitutive relation

$$\sigma = E(\alpha - 1).$$

This leads to the relation:

$$l = L - \frac{\rho g L^2}{2E}, \tag{22}$$

in a similar manner as in the previous question.

The full expression for the current length is quite messy:

$$l = \frac{1 + \left(R + S^{\star}\right)\left(\frac{1}{L_0 - R - S^{\star}} - \frac{\kappa \rho g}{2} t\right)}{\frac{1}{L_0 - R - S^{\star}} - \frac{\kappa \rho g}{2} t} - \frac{\rho g}{2E}\left(\frac{1 + \left(R + S^{\star}\right)\left(\frac{1}{L_0 - R - S^{\star}} - \frac{\kappa \rho g}{2} t\right)}{\frac{1}{L_0 - R - S^{\star}} - \frac{\kappa \rho g}{2} t}\right)^2, \tag{23}$$

however, considering the limit of $t \to \infty$, we find that:

$$l_\infty \sim (R + S^\star) - \frac{\rho g}{2E}(R + S^\star)^2 = R - \frac{\sigma^\star}{\rho g} - \frac{\rho g}{2E}\left(R - \frac{\sigma^\star}{\rho g}\right)^2. \tag{24}$$