C5.6 APPLIED COMPLEX VARIABLES

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Lecture 1a

Review - core complex analysis.

1. REVIEW

- Defns – a *region* D is a open, path-connected subset of $\mathbb{C}$.
  – a *disc* $D(a,R) = \{z : |z-a| < R\}$.
- A function $f : \mathbb{C} \to \mathbb{C}$ is differentiable at z if $\lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$ exists, in which case we write this limit as $f'(z)$. f is *holomorphic* in D if it is differentiable for each $z \in D$.
- *Cauchy-Riemann equations*. If $f(z) = u(x,y) + iv(x,y)$, where $z = x+iy$, is holomorphic, then $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.

- cross-differentiating $\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \nabla^2 u = 0$ & $\nabla^2 v = 0$, so the real & imaginary parts of a holomorphic function satisfy Laplace's equation.

$\Rightarrow$ solving Laplace's eqn in 2D is equivalent to finding a suitable holomorphic function.

- For any function $G(x,y)$, we can write $G(x,y) = g(z,\bar{z})$, where $z = x+iy$, $\bar{z} = x-iy$

$\Leftrightarrow x = \frac{z+\bar{z}}{2}$, $y = \frac{z-\bar{z}}{2i}$, and by the chain rule $\frac{\partial}{\partial z} = \frac{1}{2}\frac{\partial}{\partial x} + \frac{1}{2i}\frac{\partial}{\partial y}$, $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\frac{\partial}{\partial x} - \frac{1}{2i}\frac{\partial}{\partial y}$.

Then the Cauchy-Riemann eqns are equivalent to $\frac{\partial g}{\partial \bar{z}} = 0$, so we can think of holomorphic functions as those that are independent of $\bar{z}$.

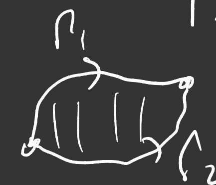
- A <u>path integral</u> along $\Gamma = \{z(t) : t \in [a,b]\}$ is defined by

$$\int_\Gamma f(z)\,dz = \int_a^b f(z(t))\, z'(t)\, dt .$$

- The <u>estimation theorem</u> provides a bound $\left|\int_\Gamma f(z)dz\right| \leq \text{length}(\Gamma) \cdot \sup_{z\in\Gamma} |f(z)|$

- <u>Cauchy's theorem</u> If $f(z)$ is holomorphic inside a simple closed contour Γ, and is continuous on Γ, then $\int_\Gamma f(z)\, dz = 0$.

An immediate corollary is the <u>deformation theorem</u> $\int_{\Gamma_1} f(z)dz = \int_{\Gamma_2} f(z)dz$ provided f is holomorphic on the region in between



- <u>Morera's theorem</u>. If $f(z)$ is continuous on D and satisfies $\int_\Gamma f(z)dz = 0$ for all contours Γ, then f is holomorphic in D.

- <u>Cauchy's integral formula</u>. Let $f(z)$ be holomorphic inside and on Γ, then for z inside Γ,

$$f(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)}{\zeta - z} \, d\zeta$$

- <u>Liouville's theorem</u> If $f(z)$ is entire (holomorphic on $\mathbb{C}$) and bounded, then it must be constant.

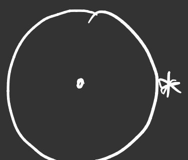
- <u>Identity theorem</u>. If $f_1(z)$ & $f_2(z)$ are holomorphic on D, and there is a sequence $z_n \in D$ with an accumulation pt in D, with $f_1(z_n) = f_2(z_n)$, then $f_1(z) = f_2(z)$ for all $z \in D$.

Lecture 16

- **Taylor's theorem** If $f(z)$ is holomorphic in a disc $D(a,R)$, then

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n \quad \text{where} \quad c_n = \frac{f^{(n)}(a)}{n!} = \frac{1}{2\pi i} \int_{\gamma(a,r)} \frac{f(z)}{(z-a)^{n+1}} \, dz.$$

- The **radius of convergence** is the largest possible value of R such that the series converges for all z in $D(a,R)$, ie the distance to the nearest singular pt. of f.



eg. $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$ has radius of convergence 1.

- **Analytic continuation**. If $f(z)$ is defined by a Taylor series, it can often be analytically continued to pts outside the radius of convergence, eg. $f(z) = \sum_{n=0}^{\infty} z^n$ converges on $|z|<1$ but is equal to $\frac{1}{1-z}$, which extends the function to be holomorphic on $\mathbb{C}\setminus\{1\}$.

- <u>Laurent's theorem</u> If $f(z)$ is holomorphic in an annulus $S < |z-a| < R$, then

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n \quad \text{where} \quad c_n = \frac{1}{2\pi i} \int_{\gamma(a,r)} \frac{f(z)}{(z-a)^{n+1}} \, dz$$

- The part of the sum with negative powers $\sum_{n=-\infty}^{-1} c_n (z-a)^n$ is called the <u>principal part</u> of f at a.

- <u>Singularities</u>. If $f(z)$ is holomorphic on punctured disc $D'(a,R)$ (i.e. $S=0$), it has an isolated singularity at a:
  - if $c_n = 0 \ \forall n < 0$, it is a <u>removable singularity</u> (eg. $\frac{\sin z}{z}$)
  - if $c_n = 0 \ \forall n < -m$ & $c_{-m} \neq 0$, it is a <u>pole of order m</u>. (eg. $\frac{1}{z^m}$)
  - if neither of these hold it is an <u>essential singularity</u> (eg. $e^{1/z}$)

- The behaviour of $f(z)$ at infinity is classified according to the behaviour of $f(\frac{1}{z})$ at $z=0$, eg. e^{z} has an essential singularity at ∞, z has a simple pole (ie. a pole of order 1) at ∞.

- The coefficient c_{-1} in the Laurent expansion is called residue of f at a; $\text{res}(f(z), a)$

- Cauchy's residue theorem. If $f(z)$ is holomorphic inside and on Γ, except at finitely many isolated singularities at $z=a_j$ inside Γ, then

$$\int_{\Gamma} f(z)\,dz = 2\pi i \sum_{j} \text{res}(f(z), a_j)$$

- Calculating residues often involves local expansions, often helped by interpreting the function as $f(z) = \frac{g(z)}{h(z)}$, where $h(z)$ has a zero at the singularity

eg. for a simple pole, $h(z) = \cancel{h(a)} + h'(a)(z-a) + \dots$, and $g(z) = g(a) + \dots$

so the residue is $\frac{g(a)}{h'(a)}$.

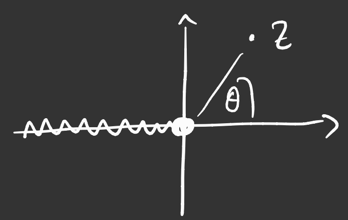
eg. $\frac{e^z}{(z-a)^3} = \frac{e^a + e^a(z-a) + \frac{1}{2}e^a(z-a)^2 + \dots}{(z-a)^3}$ so the residue at a is $\frac{1}{2}e^a$.

Lecture 2a

Review - multifunctions & contour integrals

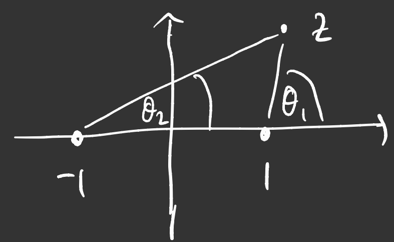
• <u>Multifunctions</u> - Some complex-valued functions (eg. $\log z$ & z^p for non-integer p) are multivalued, eg. $\log z = \log|z| + i\theta$, where $z = |z|e^{i\theta}$ can take infinitely many values depending on the choice of θ (adding 2π to θ doesn't change z, but does $\log z$). We make such functions single valued by inserting a <u>branch cut</u>, and choosing a <u>branch</u> of the function. (ie. choosing a range for θ).

eg. $\log z$, choose $\theta \in (-\pi, \pi]$



This is often called the <u>principal branch</u>, denoted $\operatorname{Log} z$.

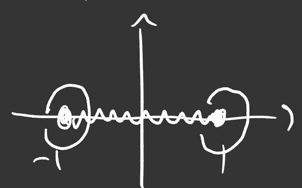
e.g. $(z^2-1)^{1/2}$. For this, write $\underbrace{(z-1)^{1/2}}_{|z-1|e^{i\theta_1}}\underbrace{(z+1)^{1/2}}_{|z+1|e^{i\theta_2}} = |z-1|^{1/2}|z+1|^{1/2}e^{i(\theta_1+\theta_2)/2}$



$z=\pm 1$ are branch pts ($z=a$ is a branch pt of $f(z)$ if on taking a circuit around a, $f(z)$ arrives at a different value).

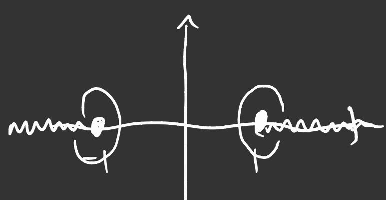
Two popular choices of branch cut:

(a)



$\theta_1, \theta_2 \in (-\pi, \pi]$

(b)

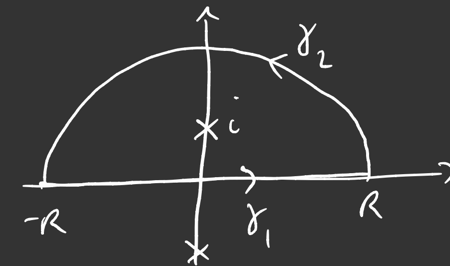


$\theta_1 \in [0, 2\pi)$, $\theta_2 \in (-\pi, \pi]$

(or $\theta_1 \in (-\pi, \pi]$, $\theta_2 \in [0, 2\pi)$)

Evaluation of integrals

eg. Calculate $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{x^2+1}\, dx$



Consider $f(z) = \dfrac{e^{iz}}{z^2+1}$ and $\Gamma = \gamma_1 \cup \gamma_2$

Note $\operatorname{res}(f, i) = \left.\dfrac{e^{iz}}{2z}\right|_i = \dfrac{e^{-1}}{2i}$, so C.R.T. $\Rightarrow \displaystyle\int_\Gamma f(z)\,dz = 2\pi i \,\frac{e^{-1}}{2i} = \frac{\pi}{e}$.

By estimation theorem, $\left|\displaystyle\int_{\gamma_2} f(z)\, dz\right| \le O\!\left(\frac{1}{R^2}\right) . \pi R \to 0$ as $R \to \infty$. (*)

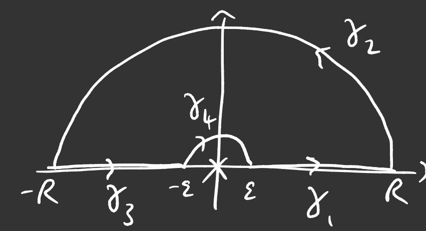
And $\displaystyle\int_{\gamma_1} f(z)\,dz \to \int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+1}\,dx$. Taking real parts $\Rightarrow$ $\boxed{\displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{x^2+1}\,dx = \frac{\pi}{e}}$

(*) More generally $\left|\displaystyle\int_{\gamma_2} F(z) e^{iz}\, dz\right| \to 0$ as $R \to \infty$ if $|F(z)| \to 0$ as $|z| \to \infty$.

(Jordan's lemma)

Lecture 2b

eg. $\int_0^\infty \frac{\sin x}{x}\,dx$. Consider $f(z) = \frac{e^{iz}}{z}$ & Γ



Cauchy's theorem $\int_\Gamma f(z)\,dz = 0$.

By Jordan's lemma $\left|\int_{\gamma_2} f(z)\,dz\right| \to 0$ as $R \to \infty$.

Also $\int_{\gamma_4} f(z)\,dz = -\pi i \,\mathrm{res}(f, 0) = -\pi i$

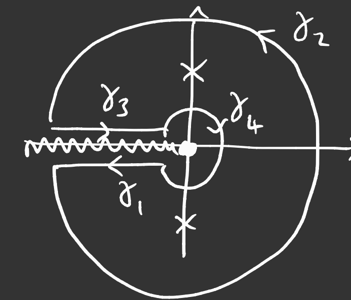
$\left[z = \varepsilon e^{i\theta}, \quad \int_\pi^0 \frac{e^{i\varepsilon e^{i\theta}}}{\cancel{\varepsilon e^{i\theta}}} \cancel{\varepsilon} i \cancel{e^{i\theta}}\,d\theta \to -\pi i \text{ as } \varepsilon \to 0 \right]$

$\int_{\gamma_1} f(z)\,dz = \int_\varepsilon^R \frac{e^{ix}}{x}\,dx$ & $\int_{\gamma_3} f(z)\,dz = \int_\varepsilon^R \frac{e^{-ix}}{-x}\,dx$, so $\int_{\gamma_1} + \int_{\gamma_3} \to \int_0^\infty \frac{e^{ix} - e^{-ix}}{x}\,dx$

Hence $0 = \int_\Gamma f(z)\,dz = \int_0^\infty \frac{2i\sin x}{x}\,dx - \pi i \Rightarrow \boxed{\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}}$

eg. $\displaystyle\int_0^\infty \frac{\log x}{x^2+4}\,dx$. Use a keyhole contour, with $f(z) = \dfrac{(\mathrm{Log}\, z)^2}{z^2+4}$

where $\mathrm{Log}\, z = \log|z| + i\theta$, where $\theta \in (-\pi, \pi]$.



Then, by C.R.T. $\displaystyle\int_\Gamma f(z)\,dz = 2\pi i\left[\operatorname{res}(f, 2i) + \operatorname{res}(f, -2i)\right]$

$$= 2\pi i\left[\frac{\left(\log 2 + \frac{i\pi}{2}\right)^2}{2.2i} + \frac{\left(\log 2 - \frac{i\pi}{2}\right)^2}{2(-2i)}\right] = 2\pi i\left[\frac{\pi \log 2}{2}\right]$$

Also $\left|\displaystyle\int_{\gamma_2} f(z)\,dz\right| \to 0$ as $R \to \infty$, and $\left|\displaystyle\int_{\gamma_4} f(z)\,dz\right| \to 0$ as $\varepsilon \to 0$.

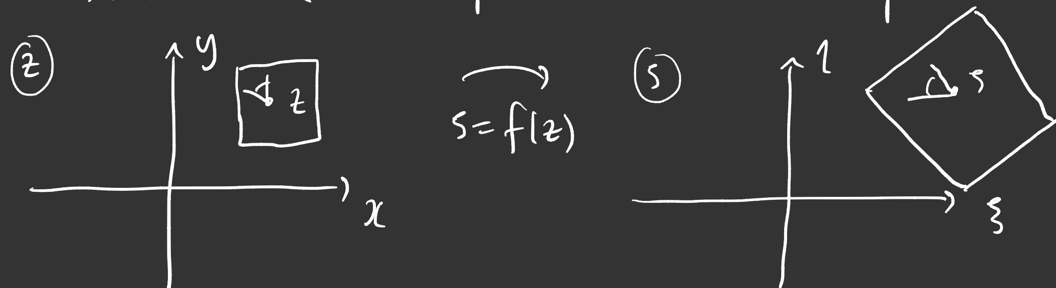
and $\displaystyle\int_{\gamma_3} f(z)\,dz = \int_\varepsilon^R \frac{(\log x + i\pi)^2}{x^2+4}\,dx$, and $\displaystyle\int_{\gamma_1} f(z)\,dz = -\int_\varepsilon^R \frac{(\log x - i\pi)^2}{x^2+4}\,dx$

Hence $\displaystyle\int_0^\infty \frac{4\pi i \log x}{x^2+4}\,dx = 2\pi i\,\frac{\pi \log 2}{2} \Rightarrow \boxed{\int_0^\infty \frac{\log x}{x^2+4}\,dx = \frac{\pi \log 2}{4}}$

Lecture 3a

Review - conformal mapping & applications.

- A holomorphic function $f(z)$ maps a point $z = x + iy$ to a new point $\zeta = f(z) = \xi + i\eta$. It maps domains in the z-plane to domains in the ζ-plane.



- $f(z)$ is conformal if $f'(z) \neq 0$
- Consider how a neighbourhood of a pt a is mapped: by Taylor's theorem

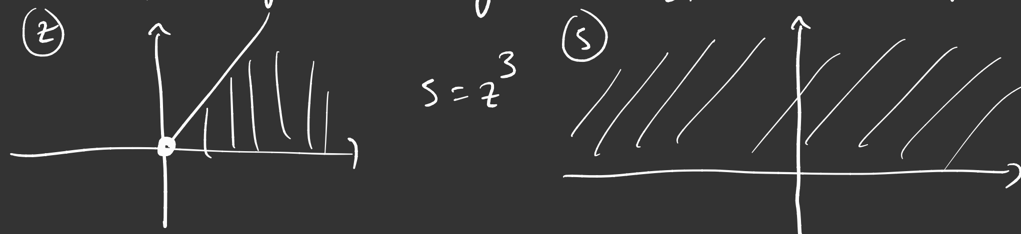
$$f(z) = f(a) + f'(a)(z-a) + \dots$$

So points that are close to a are translated and rotated/scaled by $f'(a)$.

- If two lines meet at an angle α in the z-plane, their image meets at angle α in the ζ-plane.

- Non-conformal points (critical pts) where $f'(z) = 0$ are also useful, especially on the boundary of a domain, eg. $f(z) = (z-a)^2$, maps pts of form $z = a + re^{i\theta}$ to $r^2 e^{2i\theta}$ so doubles angles at a.

eg. to map wedge $\{z : \arg z \in (0, \frac{\pi}{3})\}$ to the upper half plane



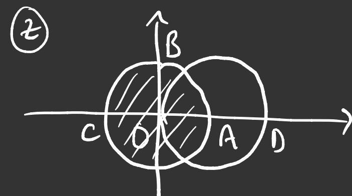
- <u>Riemann mapping theorem</u> Any simply connected domain D, except $\mathbb{C}$ itself, can be conformally mapped onto $D(0,1)$. There are three real parameters in the map.

Examples • powers of z map wedges to wedges.

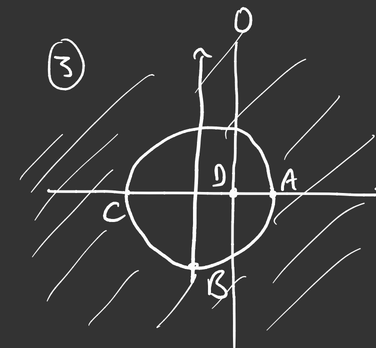
• Möbius transformation $f(z) = \dfrac{az+b}{cz+d}$ where $a,b,c,d \in \mathbb{C}$ with $ad-bc \neq 0$

map circles to circles

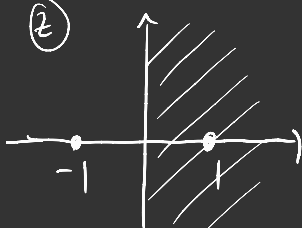
eg. $\dfrac{1}{z}$



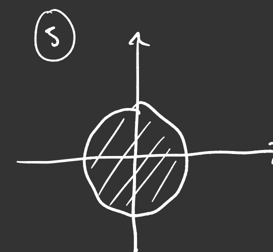
$s = \dfrac{1}{z}$



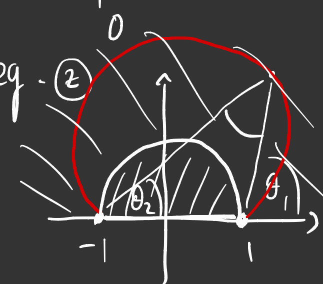
eg.



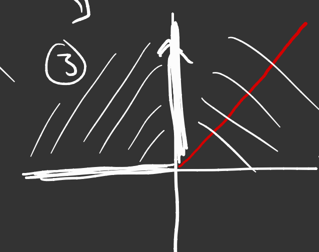
$s = \dfrac{z-1}{z+1}$



eg.

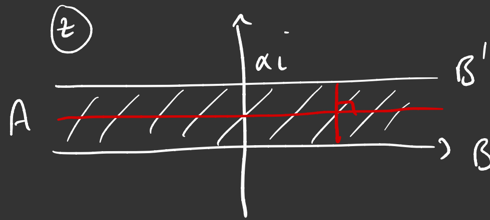


$s = \dfrac{z-1}{z+1}$



- exponentials map strips to wedges. logarithms map wedges to strips.

eg.

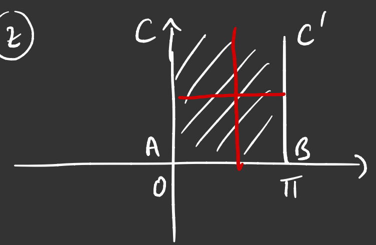


$s = e^z$

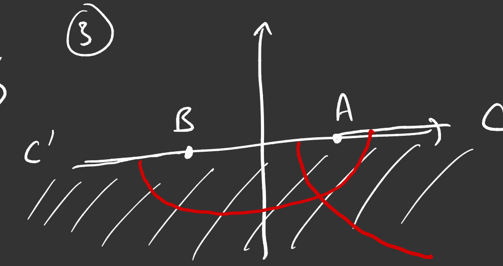
- trigonometric maps take half strips to half planes

eg. $f(z) = \cos z$

$f'(z) = -\sin z$



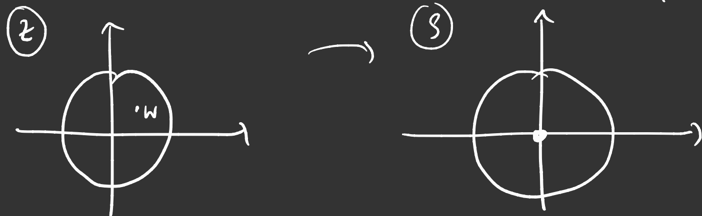
$s = \cos z$



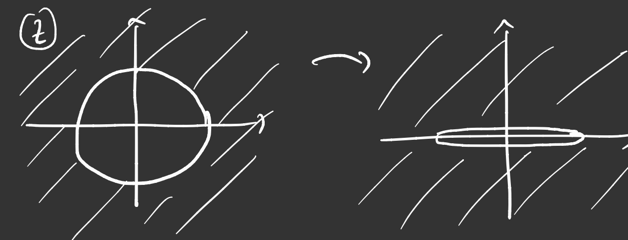
$\cos z = \cos(x+iy) = \cos x \cosh y - i \sin x \sinh y$

Lecture 3b

- two other useful mappings
  - mapping a disc to itself $\zeta = \frac{z-w}{1-\bar{w}z} e^{i\theta}$



  - Joukowski map $\zeta = \frac{1}{2}\left(z+\frac{1}{z}\right)$



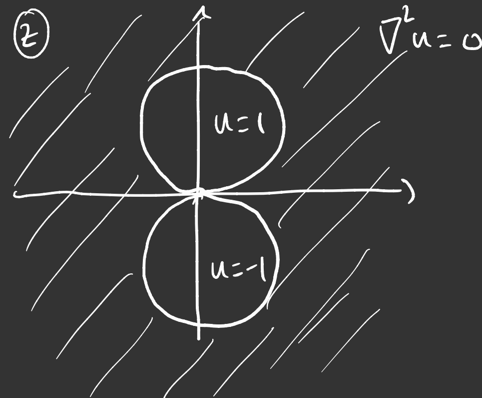
- <u>Application to steady heat flow</u>

If temperature u is governed by conduction, then in steady state it satisfies $\nabla^2 u = 0$.

[since the heat flux $\underline{q} = -k\nabla u$ (Fourier's law) satisfies $\nabla \cdot \underline{q} = 0$]

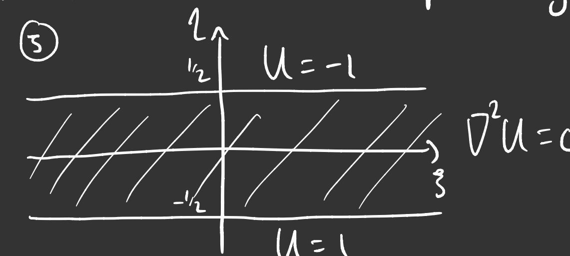
Boundary conditions are usually either fixed temp u (Dirichlet) or a fixed flux $k\frac{\partial u}{\partial n}$ (Neumann).

eg.



Find temperature u in the region exterior to the discs $|z \mp i| \leq 1$, held at $u = \pm 1$ respectively.

$$\zeta = \frac{1}{z}$$



Equivalently, find holomorphic $w(z) = u(x,y) + iv(x,y)$ satisfying $\mathrm{Re}(w) = \pm 1$ on the relevant boundaries.

Idea: map to a new domain $\zeta = f(z)$, then find $W(\zeta) = U(\xi,\eta) + iV(\xi,\eta)$ satisfying $\mathrm{Re}(W) = \pm 1$ on the mapped boundaries. Then $w(z) = W(f(z))$, and $u = \mathrm{Re}\big(W(f(z))\big)$.

In this case, by inspection, $U = -2\eta = \mathrm{Re}(2i\zeta)$, so $W(\zeta) = 2i\zeta$ works!

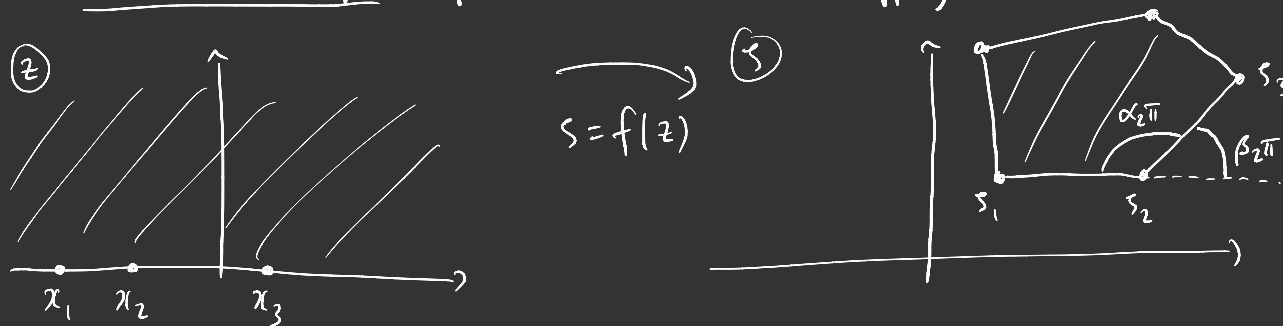
Hence $w(z) = \frac{2i}{z}$ and $u(x,y) = \mathrm{Re}\left(\frac{2i}{z}\right) = \mathrm{Re}\left(\frac{2i(x-iy)}{x^2+y^2}\right) = \boxed{\frac{2y}{x^2+y^2}}$

Lecture 4a

Schwarz-Christoffel mapping

2 FURTHER CONFORMAL MAPPING

The Schwarz-Christoffel map is a formula for mapping the Uhp to any polygon.



Label interior angles $\alpha_j\pi$ and exterior angles $\beta_j\pi$, with $\beta_j = 1-\alpha_j$. (note $\beta_j > 0$ if when traversing the boundary we turn left at ζ_j, and $\beta_j < 0$ if we turn right. Also $\sum_{j=1}^{n} \beta_j = 2$)

Note that any segment $[x, x+dx]$ is mapped to $[f(x), f(x) + f'(x)\,dx]$, ie a line segment with angle $\arg(f'(x))$. We'd like $\arg(f'(z))$ to be piecewise constant along the real axis, with jumps of $\beta_j\pi$ at each of the pre-images x_j of the vertices ζ_j.

Note that $f_j'(z) = (z - x_j)^{-\beta_j}$ has $\arg(f_j'(x)) = \begin{cases} 0 & x > x_j \\ -\beta_j\pi & x < x_j \end{cases}$

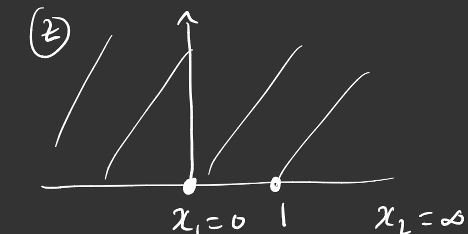
Consider $f'(z) = C \prod_{j=1}^{n} f_j'(z)$. Then $\arg(f'(x)) = \arg(C) + \sum_{j=1}^{n} \arg(f_j'(x))$, so

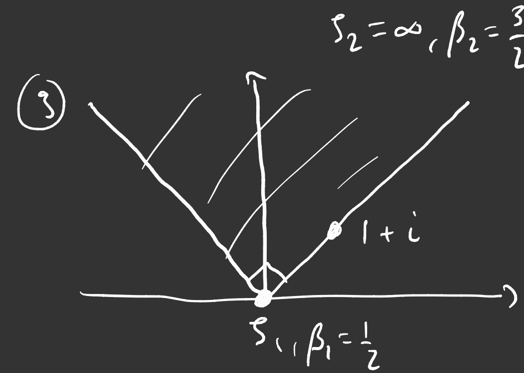
this $f'(z)$ has the properties we seek. So we can take

$$\boxed{\zeta = f(z) = A + C \int_0^z \prod_{j=1}^{n} (t - x_j)^{-\beta_j} \, dt}$$

- The constants A, C ($\in \mathbb{C}$) fix the location, size, and orientation of the polygon.
- The Riemann mapping theorem allows us to choose the pre-images x_j of 3 of the vertices, but the rest must be solved for ($f(x_j) = \zeta_j$). (Symmetry is often helpful)
- We can take one of the pre-images of the vertices to be at ∞, in which case we ignore that vertex in the product, i.e. if $x_n = \infty$, then $\prod_{j=1}^{n-1} (t - x_j)^{-\beta_j}$

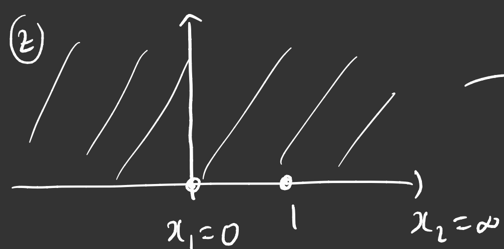
Simple examples

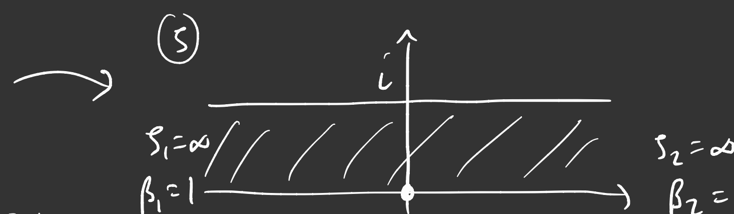




so $\zeta = A + C\int_0^z (t-0)^{-\frac{1}{2}}\,dt = A + 2Cz^{1/2}$ $\quad \left(z^{1/2} = |z|^{1/2} e^{i\theta/2} \quad \theta \in (-\pi,\pi] \right)$

$x_1 = 0 \mapsto \zeta_1 = 0 \Rightarrow A = 0$ & $1 \mapsto 1+i \Rightarrow C = \frac{1+i}{2} \Rightarrow \boxed{\zeta = \sqrt{2}\, e^{i\pi/4} z^{1/2}}$

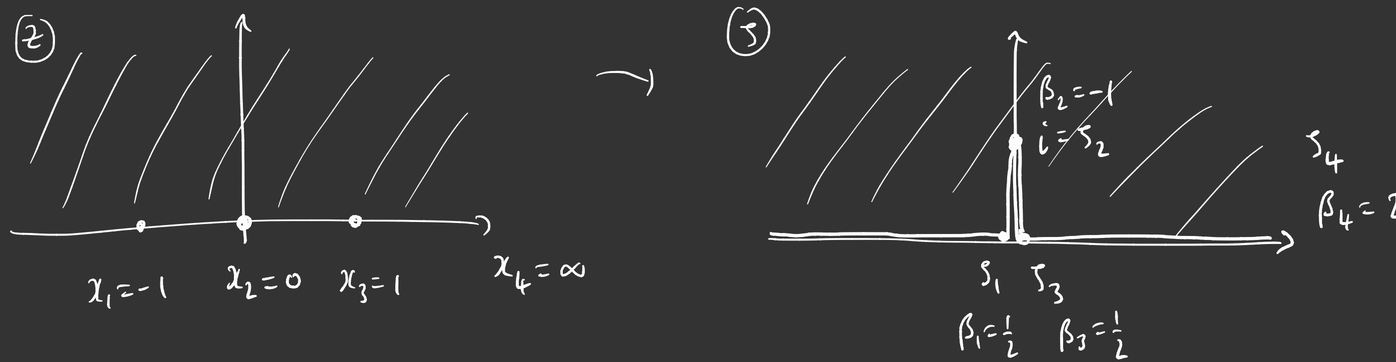




$\zeta = A + C\int_1^z (t-0)^{-1}\,dt = A + C\,\mathrm{Log}\, z$. If $1 \mapsto 0$, we need $A = 0$

Need $\mathrm{Im}(\zeta) = 0$ on +ve real axis, and $\mathrm{Im}(\zeta) = 1$ on −ve real axis, so $C = \frac{1}{\pi} \Rightarrow \boxed{\zeta = \frac{1}{\pi}\mathrm{Log}\, z}$

Lecture 4b



$$\zeta = A + C\int_0^z (t+1)^{-1/2}(t-0)^{1}(t-1)^{-1/2}\,dt = A + C\int_0^z \frac{t}{(t^2-1)^{1/2}}\,dt$$

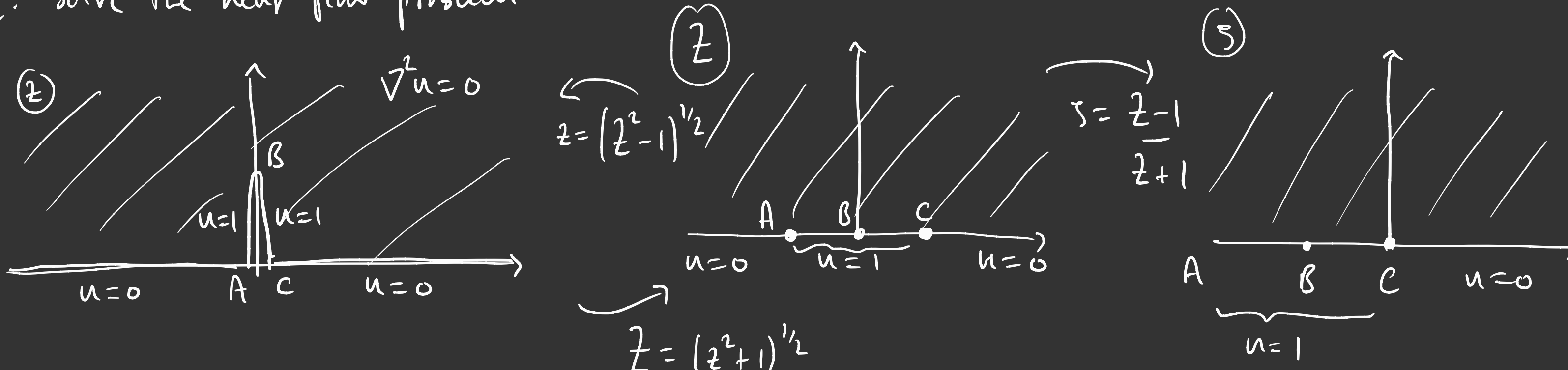
$$= A + C\,(z^2-1)^{1/2} \qquad \left[\, (z^2-1)^{1/2} = |z^2-1|^{1/2} e^{i(\theta_1+\theta_2)/2}, \text{ where } \begin{array}{l} \theta_1 = \arg(z-1) \in (-\pi,\pi] \\ \theta_2 = \arg(z+1) \in (-\pi,\pi] \end{array} \right]$$

$x_3 = 1 \mapsto \zeta_3 = 0 \quad \Rightarrow \quad A = 0.$

$x_2 = 0 \mapsto \zeta_2 = i \quad \Rightarrow \quad C = 1$

so $\boxed{\zeta = (z^2-1)^{1/2}}$

eg. Solve the heat flow problem



By inspection $u = \frac{\theta}{\pi} = \frac{1}{\pi}\operatorname{Arg}(s)$ solves Laplace's eqn & appropriate b.c. in the s-plane.

$$= \operatorname{Im}\left(\frac{1}{\pi}\operatorname{Log}(s)\right)$$

Here $\boxed{u = \operatorname{Im}\left(\frac{1}{\pi}\operatorname{Log}\left(\frac{(z^2+1)^{1/2}-1}{(z^2+1)^{1/2}+1}\right)\right)}$