Lecture 5a

Applications to fluid flow

The velocity $\underline{u}$ of an inviscid, incompressible, and irrotational fluid satisfies

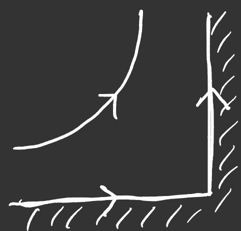
$$\nabla \cdot \underline{u} = 0 \quad \& \quad \nabla \wedge \underline{u} = \underline{0}$$

These imply the existence of a <u>velocity potential</u> ϕ and a <u>stream function</u> ψ such that (in 2D) $u = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}$ & $v = \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}$

They also imply that both ϕ & ψ satisfy Laplace's equation. In fact, these are the Cauchy-Riemann equations for $\boxed{w(z) = \phi(x,y) + i\psi(x,y)}$, the <u>complex potential</u>, which is therefore holomorphic.

Note $\frac{dw}{dz} = u - iv$, is the <u>complex velocity</u>, which is also holomorphic.

Fluid flow is in the direction of $\nabla\phi$ and perpendicular to $\nabla\psi$, so streamlines are contours of ψ.



In particular, rigid boundaries of the domain must be streamlines, so we must have

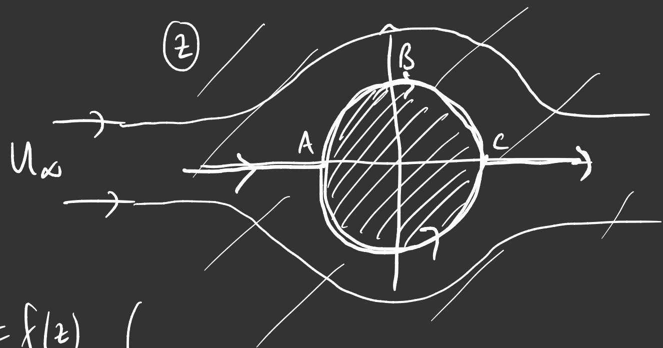
$$\boxed{\operatorname{Im} w = \psi = \text{constant on boundaries.}}$$

The difference in the value of ψ between two streamlines represents the flux of fluid flowing between them.

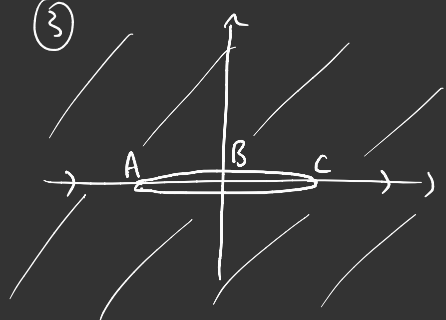
The pressure p is related to the velocity potential by Bernoulli's equation

$$\boxed{p + \tfrac{1}{2}\rho|\nabla\phi|^2 = \text{constant.}} \qquad \text{(steady flow)}$$

eg. Calculate the flow past a unit circle with velocity $(U_\infty, 0)$ at ∞.



$\zeta = f(z) = \frac{1}{2}\left(z + \frac{1}{z}\right)$



We need the x-axis, and unit circle to be a streamline, so we need $\mathrm{Im}(w) = \psi = 0$ there.

For the velocity at ∞, we need $\frac{dw}{dz} \sim U_\infty$ as $z \to \infty$

So we need $w \sim U_\infty z$ as $z \to \infty$

We need a holomorphic $W(\zeta)$ on the mapped domain, with $W \sim 2U_\infty \zeta$ as $\zeta \to \infty$, and $\mathrm{Im}(W) = 0$ on $\mathrm{Im}(\zeta) = 0$.

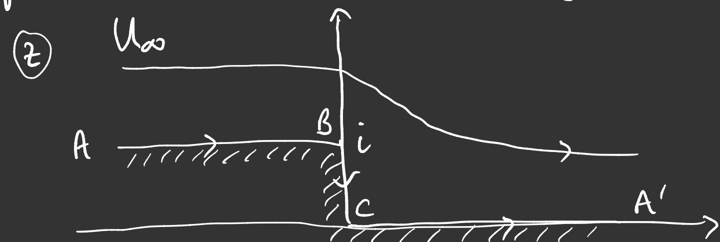
Note that $W(\zeta) = 2U_\infty \zeta$ satisfies these properties!

Hence $w(z) = W(f(z)) = U_\infty \left(z + \frac{1}{z}\right)$.

The velocity is given by $u - iv = \frac{dw}{dz} = U_\infty\left(1 - \frac{1}{z^2}\right)$

Lecture 5b

eg. flow over a backward-facing step.

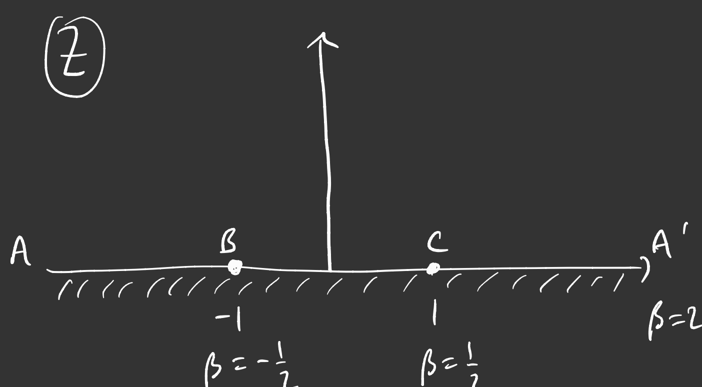


Find complex potential $w(z)$ satisfying

$\text{Im}(w) = 0$ on $ABCA'$

$w \sim U_\infty z$ as $z \to \infty$

Use Schwarz-Christoffel formula to map to this domain from the UHP.



$$z = A + C\int_1^Z (t+1)^{1/2}(t-1)^{-\frac{1}{2}}\,dt$$

$$= A + C\int_1^Z \frac{t+1}{(t^2-1)^{1/2}}\,dt$$

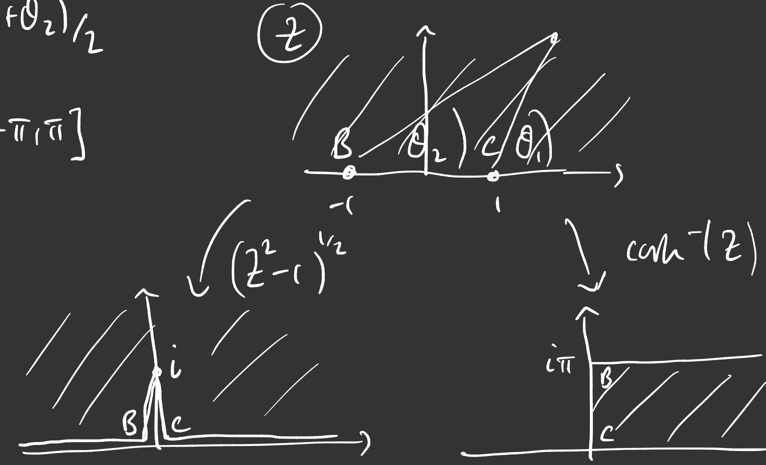
$$= A + C\left[(Z^2-1)^{1/2} + \cosh^{-1}(Z)\right]$$

Note $\cosh^{-1}(Z) = \log\left(Z + (Z^2-1)^{1/2}\right)$

Choose branch of $(z^2-1)^{1/2} = |z^2-1|^{1/2} e^{i(\theta_1+\theta_2)/2}$

with $\theta_1, \theta_2 \in (-\pi, \pi]$

& choose the principal branch of the log.



We need $1 \mapsto 0$, note $\cosh^{-1}(1) = 0$, so need $\underline{A = 0}$

We need $-1 \mapsto i$, note $\cosh^{-1}(-1) = i\pi$, so we need $C = \frac{1}{\pi}$.

Hence $\boxed{z = \frac{1}{\pi}\left[(z^2-1)^{1/2} + \cosh^{-1}(z)\right]}$

In the Z-plane, we need holomorphic $W(Z)$ satisfying $\text{Im}(W) = 0$ on $\text{Im}(Z) = 0$

and $W \sim \frac{U_\infty}{\pi} Z$ as $Z \to \infty$ (since $z \sim \frac{Z}{\pi}$ as $Z \to \infty$)

Note $W = \frac{U_\infty}{\pi} Z$ works, so $\boxed{w(z) = \frac{U_\infty}{\pi} Z(z).}$

We can write this implicitly as $z = \frac{1}{\pi}\left[\left(\frac{\pi^2 w^2}{U_\infty^2} - 1\right)^{1/2} + \cosh^{-1}\left(\frac{\pi w}{U_\infty}\right)\right]$

We can also calculate $\frac{dw}{dz} = \frac{\frac{dW}{dZ}}{\frac{dz}{dZ}} = \frac{\frac{U_\infty}{\pi}}{\frac{1}{\pi}\left(\frac{Z+1}{Z-1}\right)^{1/2}} = U_\infty\left(\frac{Z-1}{Z+1}\right)^{1/2}.$

Note that $|\underline{u}| = 0$ at C $(Z=1)$ and ∞ at B $(Z=-1)$. This is a generic property for flow at a corner with angle $\gtrless \pi$.

Lecture 6a

Steady inviscid free surface flows.

3 FREE SURFACE FLOWS

Steady inviscid free surface flows

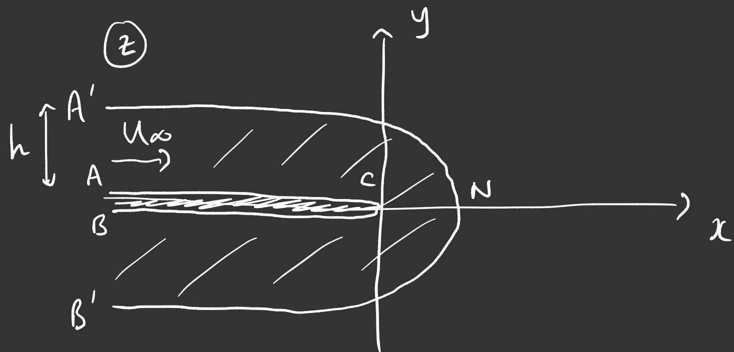
- Fixed boundaries must be streamlines, so $\psi = \operatorname{Im} w =$ constant on them.
- Free surfaces in steady flow must also be streamlines, so $\boxed{\operatorname{Im} w = \text{constant}}$ there.

They must also satisfy a dynamic condition that the pressure is a fixed constant (usually atmospheric pressure).

Bernoulli $\Rightarrow$ $p + \frac{1}{2}\rho\left|\frac{dw}{dz}\right|^2 = \text{constant}$ so we need $\boxed{\left|\frac{dw}{dz}\right| = \text{constant}}$ on free surfaces

This extra condition serves to determine the location of the free surface.

Example: teapot flow



A thin plate lies along the negative real axis.

$\Rightarrow$ fixed boundary ACB, free surface $A'NB'$

Flux between these surfaces is hU_∞, so take

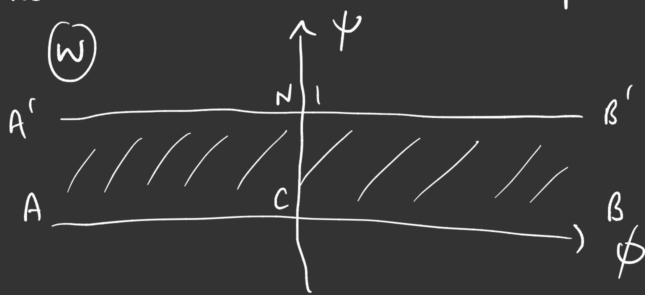
$$\psi = 0 \text{ on } ACB, \quad \psi = hU_\infty \text{ on } A'NB'$$

Also, since it's a free surface $\left|\frac{dw}{dz}\right| = U_\infty$ on $A'NB'$

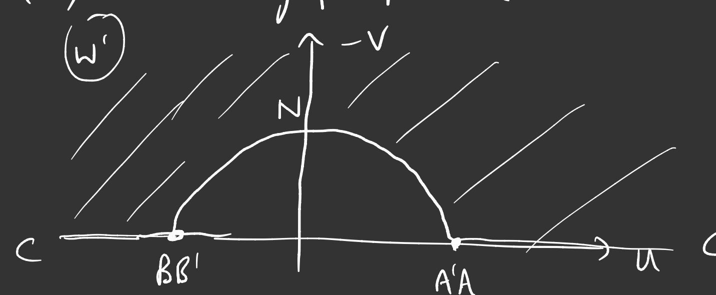
Non-dimensionalise, by scaling $z \sim h$, $w \sim hU_\infty$, then we have

$$\operatorname{Im}(w) = 0 \text{ on } ACB, \quad \operatorname{Im}(w) = 1 \text{ on } A'NB' \text{ and } |w'| = 1 \text{ on } A'NB'$$

Sketch the fluid domain in the potential plane (w) & hodograph plane (w')



$\xrightarrow{F(w)}$

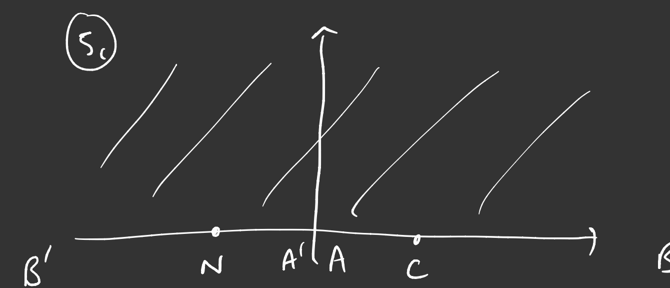


Now find a conformal mapping from the w-domain to the w'-domain, i.e. $\underline{w' = F(w)}$

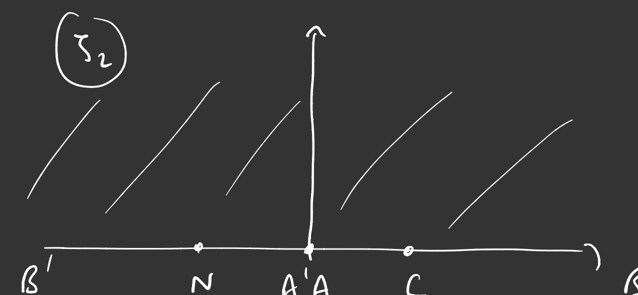
This provides an ODE that we can solve to find $w(z)$.

In practice, it's usually easier to map both w-domain and w'-domain to Uhp.

Note $\zeta_1 = e^{\pi w}$ maps w-domain to



Also $\zeta_2 = \left(\dfrac{w'-1}{w'+1}\right)^2$ maps w'-domain to



so $\zeta_1 = \zeta_2$ i.e. $\boxed{e^{\pi w} = \left(\dfrac{w'-1}{w'+1}\right)^2}$

Lecture 6b

$$e^{\pi w} = \left(\frac{w'-1}{w'+1}\right)^2 \quad \Rightarrow \text{ rearrange } \quad w' = \frac{1+e^{\pi w/2}}{1-e^{\pi w/2}} = \frac{e^{-\frac{\pi w}{4}} + e^{\frac{\pi w}{4}}}{e^{-\frac{\pi w}{4}} - e^{\frac{\pi w}{4}}}$$

$$\Rightarrow \quad \frac{\sinh \frac{\pi w}{4}}{\cosh \frac{\pi w}{4}} \frac{dw}{dz} = -1$$

we want $w=0$ at $z=0$ (part c)

$$\Rightarrow \quad \frac{4}{\pi} \ln\left(\cosh \frac{\pi w}{4}\right) = -z$$

$$\Rightarrow \quad \cosh \frac{\pi w}{4} = e^{-\pi z/4}$$

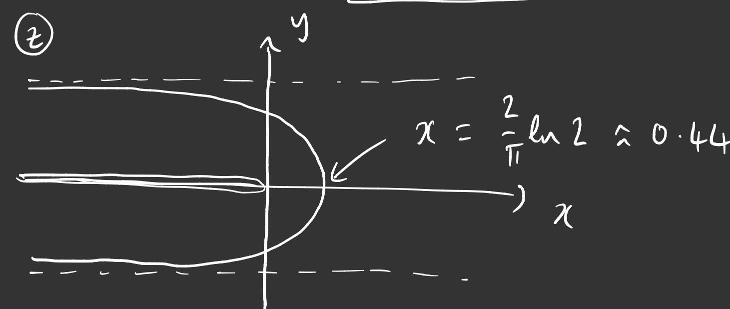
$$\boxed{w = \frac{4}{\pi} \cosh^{-1}\left(e^{-\pi z/4}\right)}$$

squaring $\Rightarrow$ $\boxed{1 + \cosh \frac{\pi w}{2} = 2e^{-\frac{\pi z}{2}}}$

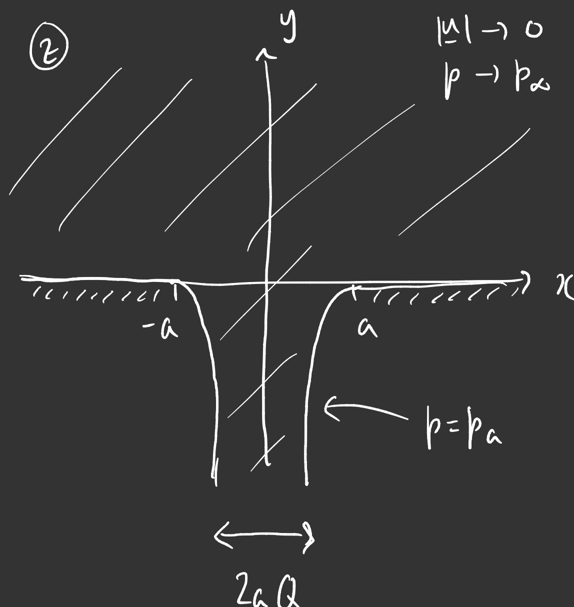
Note $\cosh \frac{\pi w}{2} = \cosh \frac{\pi \phi}{2} \cos \frac{\pi \psi}{2} + i \sinh \frac{\pi \phi}{2} \sin \frac{\pi \psi}{2}$

So on the free surface, where $\psi = 1$, we have $1 + i \sinh \frac{\pi \phi}{2} = 2e^{-\frac{\pi x}{2}} \left(\cos \frac{\pi y}{2} - i \sin \frac{\pi y}{2} \right)$

Eliminate ϕ by taking real parts $\Rightarrow$ $\boxed{1 = 2e^{-\frac{\pi x}{2}} \cos \frac{\pi y}{2}}$



Example: flow out of a slot.



$|\underline{u}| \to 0$
$p \to p_\infty$ as $z \to \infty$

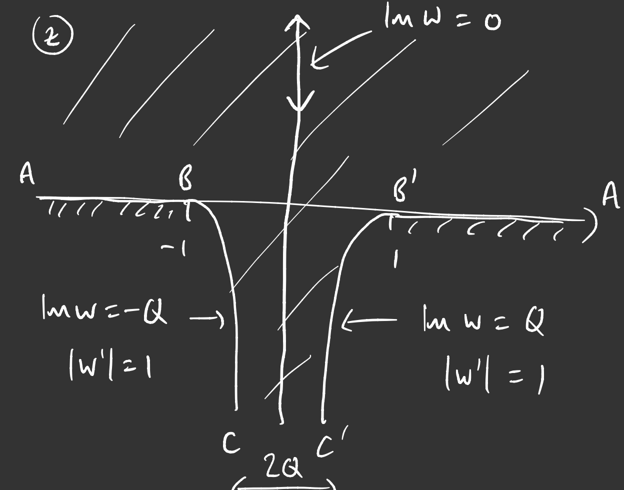
Slot of width $2a$, free surface separating tangentially at $\pm a$. Want to find the 'contraction ratio' Q.

Far-field pressure p_∞, and atmospheric pressure p_a, so

Bernoulli $\Rightarrow$ $p + \frac{1}{2}\rho U^2 = p_\infty$ $\Rightarrow$ $U = \left(\frac{2(p_\infty - p_a)}{\rho} \right)^{1/2}$

Scale the problem $z \sim a$, $w \sim aU$

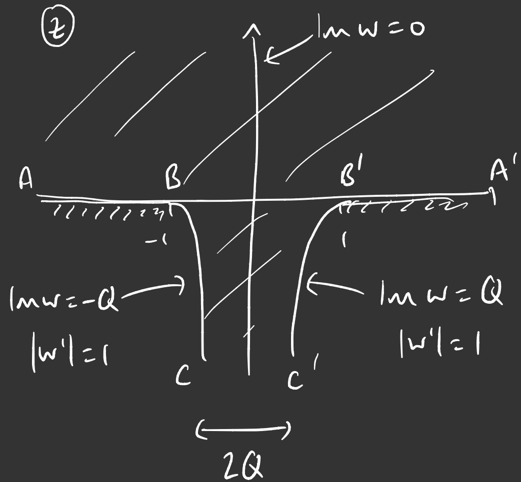
$\Rightarrow$

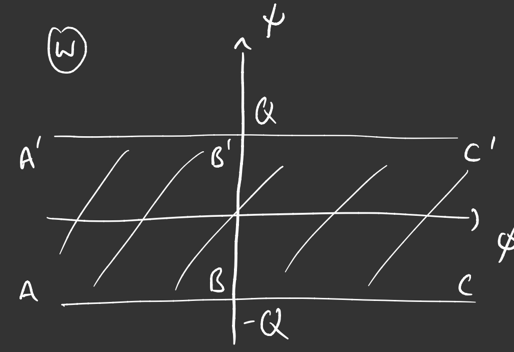


Lecture 7a

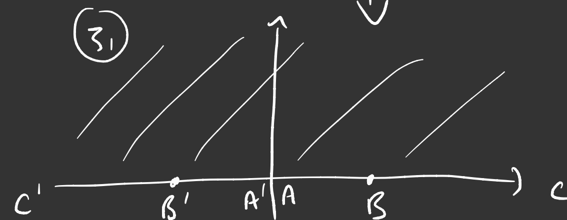
More steady inviscid free surface flows.

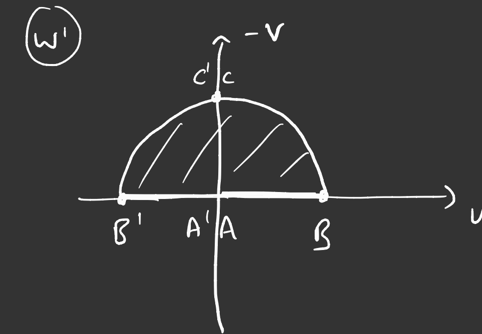
Example: Flow out of a slot.



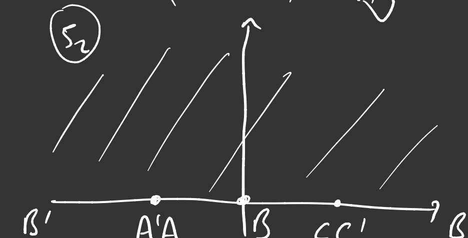


$$\zeta_1 = i e^{\frac{\pi}{2Q} w}$$





$$\zeta_2 = -\left(\frac{w'-1}{w'+1}\right)^2$$



The points are in the wrong place. But if we take $\zeta_3 = \dfrac{\zeta_1 - 1}{\zeta_1 + 1}$, then $\zeta_3 = \zeta_2$!

So we have

$$\boxed{\frac{i e^{\frac{\pi w}{2Q}} - 1}{i e^{\frac{\pi w}{2Q}} + 1} = -\left(\frac{w'-1}{w'+1}\right)^2} .$$

This gives $w' = F(w)$, in principle, but it's not easy to solve.

To find the free surface, we parameterise it using the angle θ it makes with the x-axis.

Then $w' = e^{-i\theta}$ (θ goes from 0 at B to $-\frac{\pi}{2}$ at C).

We have
$$\frac{ie^{\frac{\pi w}{2a}} - 1}{ie^{\frac{\pi w}{2a}} + 1} = -\left(\frac{e^{-i\theta} - 1}{e^{-i\theta} + 1}\right)^2 = \tan^2\frac{\theta}{2}$$

$$ie^{\frac{\pi w}{2a}} = \frac{1 + \tan^2\frac{\theta}{2}}{1 - \tan^2\frac{\theta}{2}} = \sec\theta$$

Now $\frac{dw}{d\theta} = \frac{dw}{dz}\frac{dz}{d\theta}$, so
$$\underbrace{i\frac{\pi}{2a}e^{\frac{\pi w}{2a}}}_{\frac{\pi}{2a}\sec\theta}\ \underbrace{w'}_{e^{-i\theta}}\ \frac{dz}{d\theta} = \sec\theta\tan\theta$$

$$\Rightarrow \boxed{\frac{dz}{d\theta} = \frac{2a}{\pi}\tan\theta\, e^{i\theta}}$$

Taking real & imaginary parts $\Rightarrow$ $\boxed{\dfrac{dx}{d\theta} = \dfrac{2Q}{\pi} \sin\theta \quad \& \quad \dfrac{dy}{d\theta} = \dfrac{2Q}{\pi} \dfrac{\sin^2\theta}{\cos\theta}}$

Integrating these gives $(x(\theta), y(\theta))$, a parametric representation of the free surface.

At B, we need $\theta = 0$, $x = -1$, $y = 0$

$$\Rightarrow \quad x = \frac{2Q}{\pi}(1 - \cos\theta) - 1$$

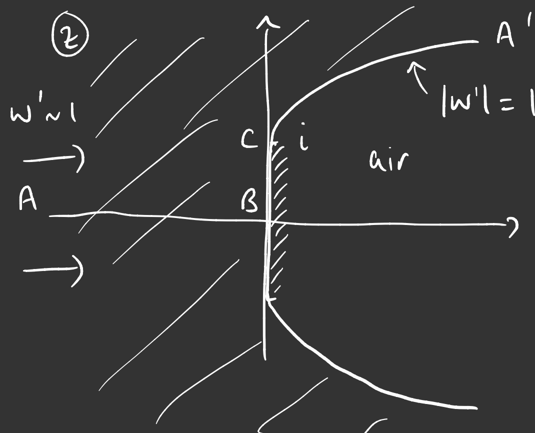
$$y = \frac{2Q}{\pi}\left(\log|\sec\theta + \tan\theta| - \sin\theta\right) \qquad \text{(check)}$$

As $\theta \to -\frac{\pi}{2}$, we should end up at C, i.e. $y \to -\infty$, $x \to -Q$

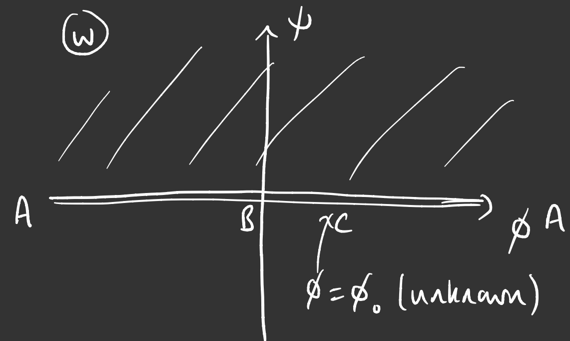
So we need $\dfrac{2Q}{\pi} - 1 = -Q \quad \Rightarrow \quad \boxed{Q = \dfrac{\pi}{\pi + 2}} \simeq 0.61$

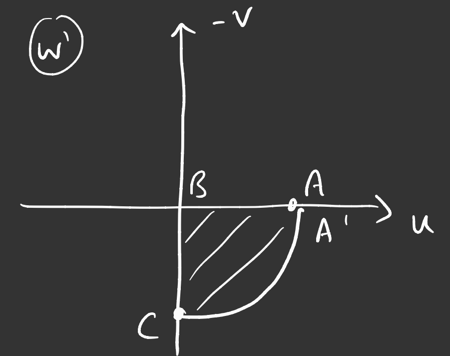
Lecture 7b

Example: separation behind a plate (Riabouchinsky flow)

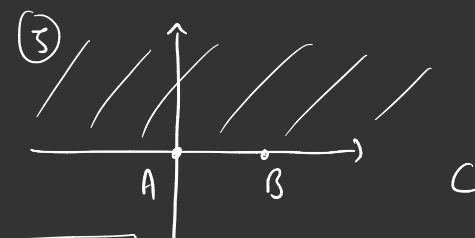


$\mathrm{Im}\, w = 0$ on $ABCA'$. Consider $\mathrm{Im}(z) > 0$ by symmetry.





$$\zeta = \left(\frac{w'^2 - 1}{w'^2 + 1}\right)^2$$ maps the w'-domain to



$w = \phi_0 \dfrac{\zeta - 1}{\zeta}$ maps this to the w-domain.

So
$$\boxed{w = \phi_0\left(1 - \left(\frac{w'^2 + 1}{w'^2 - 1}\right)^2\right) = -\phi_0 \frac{4w'^2}{(w'^2 - 1)^2}}$$

On CA', put $w' = e^{-i\theta}$ (θ goes from $\frac{\pi}{2}$ at C to 0 at A'), so $w = \dfrac{-4\phi_0 e^{-2i\theta}}{(e^{-2i\theta}-1)^2} = \dfrac{\phi_0}{\sin^2\theta}$

Then $\dfrac{dw}{d\theta} = w' \dfrac{dz}{d\theta} = -2\phi_0 \dfrac{\cos\theta}{\sin^3\theta} \implies \dfrac{dz}{d\theta} = -2\phi_0 \dfrac{\cos\theta}{\sin^3\theta} e^{i\theta}$

Taking real/imaginary parts $\implies \dfrac{dx}{d\theta} = -2\phi_0 \dfrac{\cos^2\theta}{\sin^3\theta}$, $\dfrac{dy}{d\theta} = -2\phi_0 \dfrac{\cos\theta}{\sin^2\theta}$

At C, we need $x=0$, $y=1$ at $\theta = \frac{\pi}{2}$, so integrating gives

$$x = \phi_0 \left[\cot\theta \operatorname{cosec}\theta - \log|\cot\theta + \operatorname{cosec}\theta| \right], \quad y = 2\phi_0(\operatorname{cosec}\theta - 1) + 1$$

We can see the far-field behaviour, as $\theta \to 0$, when $x \sim \dfrac{\phi_0}{\theta^2}$, $y \sim \dfrac{2\phi_0}{\theta} \implies \underline{y \sim 2\phi_0^{1/2} x^{1/2}}$

We still need to determine ϕ_0. This requires thinking about the rigid boundary BC, where we know $w = \phi$ and $w' = -iv$.

Putting these into the relationship between w and w' gives

$$\phi = \frac{4\phi_0 v^2}{(v^2+1)^2} \Rightarrow v^2 - 2\left(\frac{\phi_0}{\phi}\right)^{1/2} v + 1 = 0$$

$$\Rightarrow v = \left(\frac{\phi_0}{\phi}\right)^{1/2} - \left(\frac{\phi_0}{\phi} - 1\right)^{1/2} = \frac{\partial \phi}{\partial y}.$$

On BC, ϕ goes from 0 to ϕ_0 while y goes from 0 to 1, so integrating gives

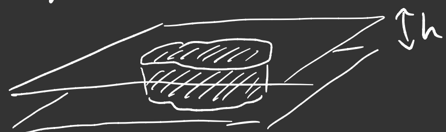
$$\int_0^{\phi_0} \frac{d\phi}{\left(\frac{\phi_0}{\phi}\right)^{1/2} - \left(\frac{\phi_0}{\phi} - 1\right)^{1/2}} = \int_0^1 dy \Rightarrow \left(2 + \frac{\pi}{2}\right)\phi_0 = 1 \Rightarrow \boxed{\phi_0 = \frac{2}{\pi + 4}}$$

Lecture 8a

Porous media flow with a free boundary

Flow in porous media

Porous flow in soil / rock is described by Darcy's law $\underline{u} = -\frac{k}{\mu}\nabla p$. Incompressibility requires $\nabla \cdot \underline{u} = 0$, so $\boxed{\nabla^2 p = 0}$.



This also describes flow in a Hele-Shaw cell, when $k = \frac{h^2}{12}$

At the edge of the fluid domain, the pressure is constant, $p = p_a = 0$. We differentiate this condition to find the kinematic condition

$$0 = \frac{Dp}{Dt} = \frac{\partial p}{\partial t} + \underline{u} \cdot \nabla p = \frac{\partial p}{\partial t} - \frac{k}{\mu}|\nabla p|^2$$

We non-dimensionalise, and write $\phi = -p$, then for a fluid domain $D(t)$, we have

$$\nabla^2 \phi = 0 \quad \text{in } D(t), \quad \text{and} \quad \phi = 0, \ \frac{\partial \phi}{\partial t} + |\nabla \phi|^2 = 0 \quad \text{on } \partial D(t).$$

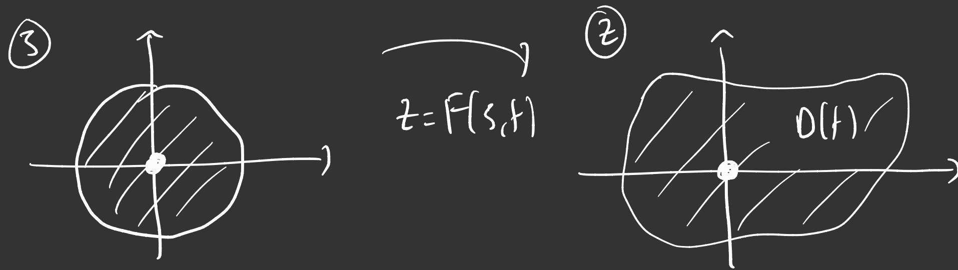
Injection/extraction at a point source

Sources/sinks are described as poles. For a source of strength Q at the origin, this means $\phi \sim \frac{Q}{2\pi} \log|z|$ as $z \to 0$. ($Q > 0$ for a source, $Q < 0$ for a sink)

Suppose $D(0)$ is given. We'd like to solve for $\phi(x,y,t)$ and for $D(t)$.

Equivalently to this, is finding a holomorphic complex potential $w(z,t) = \phi + i\psi$, with $\mathrm{Re}(w) = 0$ & $\mathrm{Re}\left(\frac{\partial w}{\partial t}\right) + \left|\frac{\partial w}{\partial z}\right|^2 = 0$ on $\partial B(t)$, & $w \sim \frac{Q}{2\pi} \log z$ as $z \to 0$.

Idea: Find a time-dependent conformal map $z = F(s,t)$ from $|s| < 1$ to $D(t)$, and find $W(s,t) = w(F(s,t),t)$.



Choose to map the origin to the origin, then we need

$$W(\zeta,t) \sim \frac{Q}{2\pi}\log\zeta \quad \text{as } \zeta \to 0.$$

& we need $\mathrm{Re}(W) = 0$ on $|\zeta| = 1$.

Note that $W(\zeta,t) = \frac{Q}{2\pi}\log\zeta$ satisfies these conditions.

For the final condition, note $W(\zeta,t) = w(F(\zeta,t),t)$, so $\frac{\partial W}{\partial t} = \frac{\partial w}{\partial z}\frac{\partial F}{\partial t} + \frac{\partial w}{\partial t}$ & $\frac{\partial W}{\partial \zeta} = \frac{\partial w}{\partial z}\frac{\partial F}{\partial \zeta}$

So we need $$\mathrm{Re}\left(\frac{\partial W}{\partial t} - \frac{\partial W}{\partial \zeta}\frac{\partial F}{\partial t}\Big/\frac{\partial F}{\partial \zeta}\right) + \left|\frac{\partial W}{\partial \zeta}\right|^2\Big/\left|\frac{\partial F}{\partial \zeta}\right|^2 = 0 \quad \text{on } |\zeta| = 1$$

But we know $\frac{\partial W}{\partial t} = 0$ & $\frac{\partial W}{\partial \zeta} = \frac{Q}{2\pi\zeta}$, so this is $\mathrm{Re}\left(-\frac{Q}{2\pi\zeta}\frac{\partial F}{\partial t}\overline{\frac{\partial F}{\partial \zeta}}\right) + \frac{Q^2}{4\pi^2|\zeta|^2} = 0$

$$\Rightarrow \boxed{\mathrm{Re}\left(\zeta\frac{\partial F}{\partial \zeta}\overline{\frac{\partial F}{\partial t}}\right) = \frac{Q}{2\pi} \quad \text{on } |\zeta| = 1}$$

Lecture 8b

The kinematic condition has become $\mathrm{Re}\left(\zeta \frac{\partial F}{\partial \zeta} \overline{\frac{\partial F}{\partial t}} \right) = \frac{Q}{2\pi}$ on $|\zeta| = 1$. (✱)

The initial domain $D(0)$ determines $F(\zeta, 0)$, the (✱) determines how $F(\zeta, t)$ evolves.

In practice, we make some assumption about the form of $F(\zeta, t)$, eg. it's a polynomial – then (✱) provides ODEs that show how the coefficients evolve through time.

Eg. $F(\zeta, t) = R(t)\zeta$ (ie. $\partial D(t)$ is a circle of radius $R(t)$).

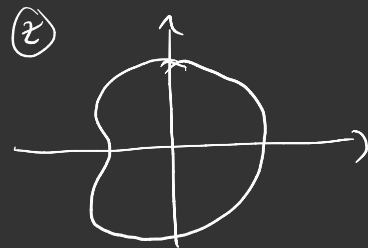
Then (✱) becomes $\mathrm{Re}\left(\zeta R \dot{R} \bar{\zeta} \right) = \frac{Q}{2\pi}$ on $|\zeta| = 1$

$$\Rightarrow \quad R\dot{R} = \frac{Q}{2\pi}$$

$$\Rightarrow \quad \pi R^2 = \pi R_0^2 + Qt$$

This is hopefully as expected from consideration of global mass conservation.

Eg. Suppose $F(\zeta,0) = a_{10}\zeta + a_{20}\zeta^2$, where a_{10} & a_{20} are real, positive constants, with $a_{10} > 2a_{20}$.



On $\zeta = e^{i\theta}$, $x = a_{10}\cos\theta + a_{20}\cos 2\theta$

$y = a_{10}\sin\theta + a_{20}\sin 2\theta$

Now assume $F(\zeta,t) = a_1(t)\zeta + a_2(t)\zeta^2$. Then ⊛ gives

$$\mathrm{Re}\left(\zeta\left(a_1 + 2a_2\zeta\right)\left(\dot{a}_1\bar{\zeta} + \dot{a}_2\bar{\zeta}^2\right)\right) = \frac{Q}{2\pi} \quad \text{on } |\zeta| = 1$$

$$\Rightarrow \mathrm{Re}\left(a_1\dot{a}_1 + 2a_2\dot{a}_2 + 2a_2\dot{a}_1\zeta + a_1\dot{a}_2\bar{\zeta}\right) = \frac{Q}{2\pi} \quad \text{on } \zeta = e^{i\theta}$$

$$\left(a_1\dot{a}_1 + 2a_2\dot{a}_2\right) + \left(2a_2\dot{a}_1 + a_1\dot{a}_2\right)\cos\theta = \frac{Q}{2\pi}$$

Since this must hold for all θ, we have

$$\boxed{a_1\dot{a}_1 + 2a_2\dot{a}_2 = \frac{Q}{2\pi} \quad (1) \qquad \& \qquad 2a_2\dot{a}_1 + a_1\dot{a}_2 = 0 \quad (2)}$$

$(1) \Rightarrow a_1^2 + 2a_2^2 = \frac{Q}{\pi}t + a_{10}^2 + 2a_{20}^2$ (using initial condition)

$(2) \Rightarrow a_1^2 a_2 = a_{10}^2 a_{20}$

Alternatively, eliminating $\dot{a}_2$ from (1) $\Rightarrow$ $a_1\dot{a}_1 \underbrace{\left(1 - \frac{4a_2^2}{a_1^2}\right)}_{>0 \text{ initially.}} = \frac{Q}{2\pi}$

If $Q>0$, a_1 increases in time, a_2 decreases
$\Rightarrow$ domain tends to a circle

If $Q<0$, a_1 decreases, a_2 increases
$\Rightarrow$ domain becomes more asymmetric, and a cusp develops when $a_1 = 2a_2$ $\left(\frac{\partial F}{\partial \zeta} = 0\right)$