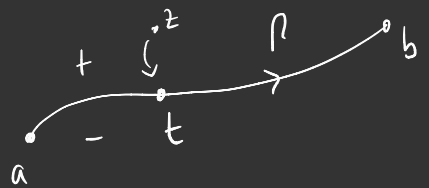
Lecture 9a

Plemelj formulae

4. PLEMELJ FORMULAE & APPLICATIONS



We consider holomorphic functions defined by a Cauchy integral

$$\boxed{w(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(s)}{s-z}\,ds}$$

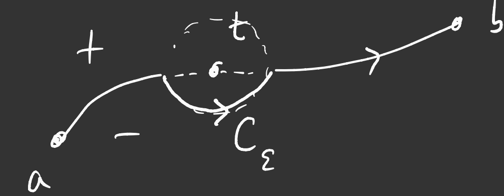
where $\Gamma = \{z(t) \in \mathbb{C} : t_0 < t < t_1,\ z(t_0) = a,\ z(t_1) = b\}$, and we write $\bar{\Gamma} = \Gamma \cup \{a,b\}$.

We are interested in the limiting values of $w(z)$ as z approaches Γ from either side at t.

[We assume f is sufficiently smooth that it can be extended to be holomorphic in a neighbourhood of t]

We define the $+/-$ sides of Γ as the left/right sides in the direction of integration.

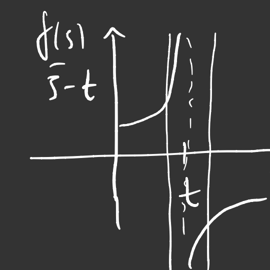
As $z \to t^+$, deform Γ by deleting the section $\gamma_\varepsilon = \Gamma \cap D(t, \varepsilon)$ and replacing with C_ε, the semi-circular arc.



Then $$w_+(t) = \frac{1}{2\pi i}\left(\int_{C_\varepsilon} + \int_{\Gamma \setminus \gamma_\varepsilon}\right) \frac{f(s)}{s-t}\, ds$$

$$\to \frac{1}{2} f(t) + \frac{1}{2\pi i} ⨍_\Gamma \frac{f(s)}{s-t}\, ds \qquad \left[s = t + \varepsilon e^{i\theta}\right]$$

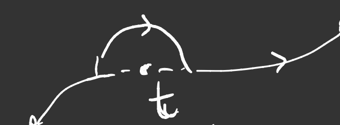
where $⨍_\Gamma$ is the principal value integral defined by $\lim_{\varepsilon \to 0} \int_{\Gamma \setminus \gamma_\varepsilon} \frac{f(s)}{s-t}\, ds$



which exists because the logarithmic singularities from $s = t \pm \varepsilon$ cancel out.

As $z \to t^-$, deform Γ the other way, so

$$w_-(t) = -\frac{1}{2} f(t) + \frac{1}{2\pi i} ⨍ \frac{f(s)}{s-t}\, ds.$$



[minus sign, since we go backwards around semicircle]

Together, we have $\boxed{W_\pm(t) = \pm \frac{1}{2} f(t) + \frac{1}{2\pi i} \fint_\Gamma \frac{f(s)}{s-t} \, ds}$. These are the <u>Plemelj formulae</u>.

At the end points a, b, we'll make use of the following results:

- if $f(s) \to 0$ as $s \to a$, then $W(z) = O(1)$ as $z \to a$.
- if $f(s) = O(1)$ as $s \to a$, then $W(z) = O(\log(z-a))$ as $z \to a$.
- if $f(s) = O((s-a)^{-\alpha})$ for $\alpha \in (0,1)$, then $W(z) = O((z-a)^{-\alpha})$ as $z \to a$.

Lecture 9b

<u>Example</u> (Problem 1) We want to find $w(z)$, holomorphic on $\mathbb{C}\setminus\bar{\Gamma}$, such that the limiting values as $z \to t \in \Gamma$ satisfy $\underline{w_+(t) - w_-(t) = G(t)}$, where G is a prescribed continuous function.

Seek a solution as a Cauchy integral $w(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(s)}{s-z}\,ds$.

Then Plemelj $\Rightarrow$ $w_+(t) + w_-(t) = \frac{1}{\pi i} ⨍ \frac{f(s)}{s-t}\,ds$ & $w_+(t) - w_-(t) = f(t)$.

So we can simply read off that we need to take $f(s) = G(s)$. Hence we can write

$$\boxed{w(z) = \frac{1}{2\pi i}\int_\Gamma \frac{G(s)}{s-z}\,ds + h(z)}$$

where $h(z)$ is holomorphic on $\mathbb{C}\setminus\bar{\Gamma}$ and continuous across Γ. In fact, using Morera's theorem, such a function must be holomorphic on $\mathbb{C}\setminus\{a,b\}$.

Determining $h(z)$ requires additionally specifying the behaviour of $w(z)$ at the end points a, b of Γ, and as $z \to \infty$. For example, if $w(z)$ has at worst logarithmic singularities at a/b, and $w \to 0$ as $z \to \infty$, then since h has only isolated singularities at a/b, it is in fact holomorphic everywhere, and by Liouville's theorem it must be zero.

Example (Problem 2) We specialise to the case when $\Gamma = (0, c)$, a section of the real axis, and suppose $\mathrm{Im}(w)$ is specified as $y \to 0^{\pm}$. ie. $\mathrm{Im}(w_{\pm}) = g_{\pm}(x)$ for prescribed $g_{\pm}(x)$.

Again, seek a solution $w(z) = \frac{1}{2\pi i} \int_0^c \frac{f(\xi)}{\xi - z} \, d\xi$

Plemelj formulae $\Rightarrow$ $w_+(x) - w_-(x) = f(x)$ & $w_+(x) + w_-(x) = \frac{1}{\pi i} ⨍_0^c \frac{f(\xi)}{\xi - x} d\xi$.

<u>Case I</u> $g_+ = -g_-$. Then $\mathrm{Im}(w_+ + w_-) = 0$, so f must be purely imaginary.

So $w_+ - w_-$ is purely imaginary, so is equal to $2ig_+$. $\Rightarrow f(x) = 2ig_+(x)$

Hence $w(z) = \frac{1}{\pi} \int_0^c \frac{g_+(\xi)}{\xi - z} d\xi + h(z)$

where $h(z)$ is holomorphic on $\mathbb{C} \setminus [0,c]$, with $\mathrm{Im}(h) = 0$ on $(0,c)$.

<u>Case II</u> $g_+ = g_-$. Then $\mathrm{Im}(w_+ - w_-) = 0$, so f must be purely real.

So $w_+ + w_-$ is purely imaginary, and therefore equal to $2ig_+$.

Hence $f(x)$ must satisfy $\boxed{\frac{1}{\pi} ⨍_0^c \frac{f(\xi)}{\xi - x} d\xi = -2g_+(x)}$ for $x \in (0,c)$

This is a singular integral equation (S.I.E) for $f(x)$.

Lecture 10a

Applications of Plemelj formulae

Problem 2, Case II We are seeking $w(z)$, holomorphic on $\mathbb{C}\setminus\bar{\Gamma}$, where $\Gamma = (0,c)$,

with $\operatorname{Im}(w_\pm) = g_\pm$ on $(0,c)$, with $g_+ = g_-$.

We write $w(z) = \frac{1}{2\pi i}\int_0^c \frac{f(\xi)}{\xi - z}\,d\xi$, and we found $w_+ - w_- = f$ is purely real,

and so $w_+ + w_- = 2ig_+(x) = \frac{1}{\pi i} ⨍_0^c \frac{f(\xi)}{\xi - x}\,d\xi$. $(*)$

Idea: Suppose we can find $\tilde{w}(z)$, holomorphic on $\mathbb{C}\setminus\bar{\Gamma}$, with $\tilde{w}_+(x) + \tilde{w}_-(x) = 0$ on Γ.

Then consider $W(z) = \dfrac{w(z)}{\tilde{w}(z)}$, which has

$$W_+(x) - W_-(x) = \frac{w_+(x)}{\tilde{w}_+(x)} - \frac{w_-(x)}{\tilde{w}_-(x)} = \frac{w_+(x) + w_-(x)}{\tilde{w}_+(x)} = \frac{2ig_+(x)}{\tilde{w}_+(x)} \quad (1)$$

$$W_+(x) + W_-(x) = \frac{w_+(x) - w_-(x)}{\tilde{w}_+(x)} = \frac{f(x)}{\tilde{w}_+(x)} \quad (2)$$

① is a version of problem 1; we can write

$$W(z) = \frac{1}{2\pi i}\int_0^c \frac{F(\xi)}{\xi - z}\, d\xi + H(z)$$ where H is holomorphic $\mathbb{C}\setminus\{0,c\}$.

and read off that $F(x) = \dfrac{2i\, g_+(x)}{\tilde{w}_+(x)}$.

Then $$\boxed{w(z) = \tilde{w}(z) W(z) = \tilde{w}(z)\left[\frac{1}{\pi}\int_0^c \frac{g_+(\xi)}{\tilde{w}_+(\xi)(\xi - z)}\, d\xi + H(z)\right]}$$

Note that $W_+(x) + W_-(x) = \dfrac{1}{\pi i}\fint_0^c \dfrac{F(\xi)}{\xi - x}\, d\xi + 2H(x) = \dfrac{f(x)}{\tilde{w}_+(x)}$ from ②.

so $$f(x) = \tilde{w}_+(x)\left[\frac{2}{\pi}\fint_0^c \frac{g_+(\xi)}{\tilde{w}_+(\xi)(\xi - x)}\, d\xi + 2H(x)\right]$$ satisfies the S.I.E. (*).

How do we find $\tilde{w}(z)$? By inspection, $\hat{w}(z) = z^{1/2}(c-z)^{1/2}$, with branch cut on $[0,c]$.

Then $\tilde{w}_\pm(x) = \pm x^{1/2}(c-x)^{1/2}$, so this has the desired properties!

More deductively, note we need $\dfrac{\tilde{w}_+(x)}{\tilde{w}_-(x)} = -1$, so $\log \tilde{w}_+ - \log \tilde{w}_- = \pi i + 2m\pi i \quad m \in \mathbb{Z}$.

This is again a version of problem 1, with solution

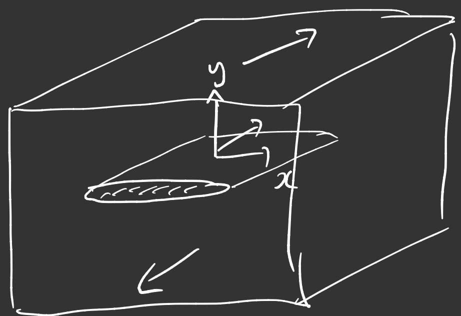
$$\log \tilde{w}(z) = \frac{1}{2\pi i} \int_0^c \frac{\pi i + 2m\pi i}{\xi - z} \, d\xi + \tilde{h}(z) \qquad \Rightarrow \quad \tilde{w}(z) = h^*(z) \left(\frac{c-z}{z} \right)^{m+\frac{1}{2}}$$

$$= \left(m+\tfrac{1}{2}\right)\left[\log(c-z) - \log(-z) \right] + \tilde{h}(z)$$

where h^* is holomorphic on $\mathbb{C} \setminus \{0, c\}$.

Lecture 10b

Example from fracture mechanics



The elastic displacement field in antiplane strain around a Mode III crack is $(0,0,\Phi)$, where $\nabla^2\Phi = 0$, with

$$\frac{\partial \Phi}{\partial y} \to T \text{ as } y \to \pm\infty, \qquad \frac{\partial \Phi}{\partial y} = 0 \text{ on } y=0,\ 0<x<c.$$

We also require $|\nabla\Phi|$ has an inverse square-root singularity at $x=0$ and c. (related to the stress intensity factor)

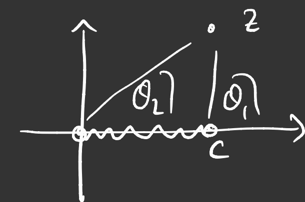
We write $\Phi = Ty - \phi$, and then write $\frac{\partial \phi}{\partial y} = \operatorname{Im}(w)$. Then we need w to be holomorphic on $\mathbb{C}\setminus[0,c]$, with $\operatorname{Im}(w) = T$ on $(0,c)$, and with $w = O(z^{-1/2})$ as $z \to 0$, $w = O((z-c)^{-1/2})$ as $z \to c$, and $w = O(1/z^2)$ as $z \to \infty$.

This is Case II, which we have solved.

We choose $\tilde{w}(z) = z^{-1/2}(c-z)^{-1/2}$ (ie. incorporating the desired singularities for w).

$$z^{1/2}(c-z)^{1/2} = -i z^{1/2}(z-c)^{1/2} = -i|z|^{1/2}|z-c|^{1/2}e^{i(\theta_1+\theta_2)/2}$$

with $\theta_1, \theta_2 \in [-\pi, \pi]$

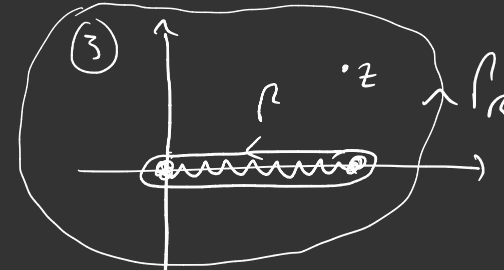


So $\tilde{w}_\pm(x) = \pm x^{-1/2}(c-x)^{-1/2}$

Then, from earlier,
$$w(z) = \frac{1}{z^{1/2}(c-z)^{1/2}}\left[\frac{1}{\pi}\int_0^c \frac{T\xi^{1/2}(c-\xi)^{1/2}}{\xi - z}\,d\xi + H(z)\right]$$

H cannot have poles at 0 or c, else w would have singularities worse than inverse square-root singularities, so H must be entire. But we also need $H = O(1/z)$ as $z \to \infty$, so H is bounded and Liouville $\Rightarrow H \equiv 0$.

We can calculate the integral using a contour.



Define $f(\zeta) = \dfrac{\zeta^{1/2}(c-\zeta)^{1/2}}{\zeta - z}$.

Then $I = \displaystyle\int_0^c \frac{\xi^{1/2}(c-\xi)^{1/2}}{\xi - z}\, d\xi = -\frac{1}{2}\int_{\Gamma} f(\zeta)\, d\zeta = -\frac{1}{2}\int_{\Gamma_R} f(\zeta)\, d\zeta + \pi i \,\mathrm{res}(f, \zeta = z)$

Note, for $\zeta = Re^{i\theta}$, $\zeta^{1/2}(c-\zeta)^{1/2} = -i\zeta^{1/2}(\zeta-c)^{1/2} = -i\zeta\left(1-\frac{c}{\zeta}\right)^{1/2} = -i\zeta\left(1-\frac{c}{2\zeta}+\dots\right)$

$$\frac{1}{\zeta - z} = \frac{1}{\zeta}\left(1-\frac{z}{\zeta}\right)^{-1} = \frac{1}{\zeta}\left(1+\frac{z}{\zeta}+\dots\right)$$

so $f(\zeta) = -i\left(1 + \left(z-\frac{c}{2}\right)\frac{1}{\zeta} + O\left(\frac{1}{\zeta^2}\right)\right)$

$$\Rightarrow \int_{\Gamma_R} f(\zeta)\, d\zeta = \int_0^{2\pi} -i\left[1 + \left(z-\frac{c}{2}\right)\frac{1}{Re^{i\theta}} + O\left(\frac{1}{R^2}\right)\right] Rie^{i\theta}\, d\theta \to 2\pi\left(z-\frac{c}{2}\right)$$

as $R \to \infty$.

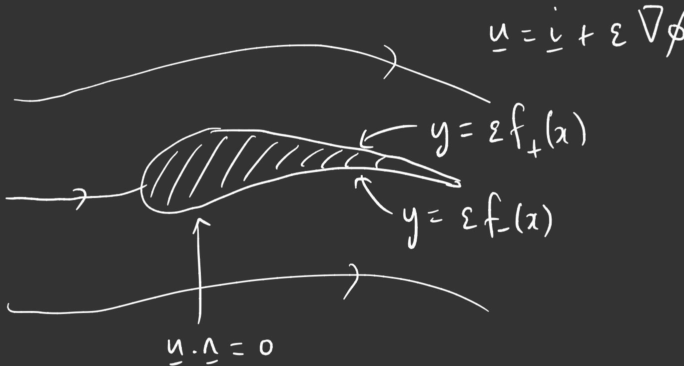
so $I = -\pi\left(z-\frac{c}{2}\right) + \pi i z^{1/2}(c-z)^{1/2}$,

& finally, $w(z) = \dfrac{1}{z^{1/2}(c-z)^{1/2}}\dfrac{T}{\pi} I = \boxed{Ti - \dfrac{T\left(z-\frac{c}{2}\right)}{z^{1/2}(c-z)^{1/2}}}$

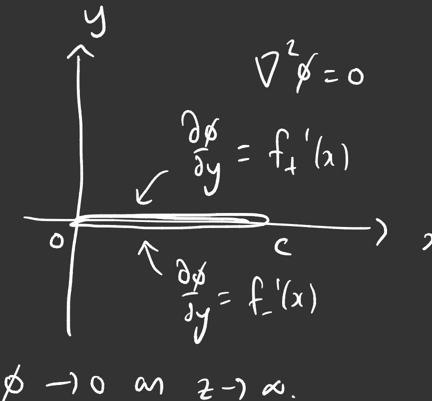
Lecture 11a

More applications of Plemelj formulae & Riemann-Hilbert problems

Example from aerofoil theory



linearise $\longrightarrow$



The Kutta condition says that $\nabla \phi$ must be finite at the trailing edge $z = c$, but we expect an inverse square-root singularity at the leading edge $z = 0$.

We write $w = -\left(\frac{\partial \phi}{\partial x} - i \frac{\partial \phi}{\partial y}\right)$, $\Gamma = (0, c)$, and $g_\pm(x) = f_\pm'(x)$, then we need:

w holomorphic on $\mathbb{C} \setminus \overline{\Gamma}$, with $\mathrm{Im}(w_\pm) = g_\pm(x)$, and $w \to 0$ as $z \to \infty$, and $w = O(z^{-1/2})$ as $z \to 0$ and $w = O(1)$ as $z \to c$.

We consider the case $g_+ = g_- = g$, which represents a thin aerofoil 

This is case II of problem 2 from the last lecture.

Given the desired singularities for w, a good choice for $\hat{w}(z)$ is $\left(\frac{c-z}{z}\right)^{1/2}$.

Instead, we'll try $\hat{w}(z) = z^{-1/2}(c-z)^{-1/2}$. Then

$$w(z) = \frac{1}{z^{1/2}(c-z)^{1/2}}\left[\frac{1}{\pi}\int_0^c \frac{g(\xi)\,\xi^{1/2}(c-\xi)^{1/2}}{\xi - z}\,d\xi + H(z)\right]$$

H could have singularities at 0 or c, but they would make w too singular there, so we need H to be entire. In order for $w \to 0$ as $z \to \infty$, we need H to be bounded, so Liouville $\Rightarrow$ H is constant.

In order to make w finite at $z = c$, we need to choose $H = -\frac{1}{\pi}\int_0^c \frac{g(\xi)\,\xi^{1/2}(c-\xi)^{1/2}}{\xi - c}\,d\xi$.

So $$w(z) = \frac{1}{z^{1/2}(c-z)^{1/2}} \frac{1}{\pi} \int_0^c g(\xi)\, \xi^{1/2} (c-\xi)^{1/2} \underbrace{\left[\frac{1}{\xi - z} - \frac{1}{\xi - c} \right]} \, d\xi$$

$$\frac{z-c}{(\xi-z)(\xi-c)} = \frac{c-z}{(\xi-z)(c-\xi)}$$

$$= \boxed{\left(\frac{c-z}{z}\right)^{1/2} \frac{1}{\pi} \int_0^c g(\xi) \left(\frac{\xi}{c-\xi}\right)^{1/2} \frac{1}{\xi - z} \, d\xi}$$

This is the same as we would have obtained taking $\tilde{w}(z) = \left(\frac{c-z}{z}\right)^{1/2}$.

Lecture 11b

General Riemann - Hilbert problem

Recall the Plemelj formulae for $w(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(s)}{s-z}\,ds$ are $W_\pm(z) = \pm\frac{1}{2}f(z) + \frac{1}{2\pi i}⨍_\Gamma \frac{f(s)}{s-z}\,ds$

Suppose we seek $w(z)$ holomorphic on $\mathbb{C}\setminus\bar{\Gamma}$, with

$$a(z)\,w_+(z) + b(z)\,w_-(z) = c(z) \quad \text{on } \Gamma$$

First step is to introduce $\tilde{w}(z)$ satisfying $a(z)\,\tilde{w}_+(z) + b(z)\,\tilde{w}_-(z) = 0$ on Γ.

How? Rearrange as $\frac{\tilde{w}_+}{\tilde{w}_-} = -\frac{b}{a}$, then take logs $\Rightarrow$ $\log\tilde{w}_+ - \log\tilde{w}_- = \log\left(-\frac{b}{a}\right)$.

so $$\log\tilde{w}(z) = \frac{1}{2\pi i}\int_\Gamma \log\left(-\frac{b(s)}{a(s)}\right)\frac{ds}{s-z}.$$

Then consider $W(z) = \dfrac{w(z)}{\tilde{w}(z)}$, which satisfies $W_+(z) - W_-(z) = \dfrac{c(z)}{a(z)\,\tilde{w}_+(z)}$ on Γ.

Hence $\dfrac{w(z)}{\tilde{w}(z)} = W(z) = \dfrac{1}{2\pi i}\displaystyle\int_\Gamma \frac{c(s)}{a(s)\,\tilde{w}_+(s)}\,\frac{ds}{s-z} + H(z)$

where H must be holomorphic except possibly at the end points of Γ.

This also gives us a method for solving a more general singular integral equation of the form

$$a(z)\,f(z) + b(z) \fint_\Gamma \frac{f(s)}{s-z}\,ds = c(z) \quad \text{on } \Gamma$$

By defining $w(z) = \dfrac{1}{2\pi i}\displaystyle\int_\Gamma \frac{f(s)}{s-z}\,ds$, we can re-write this as a Riemann-Hilbert problem.

Lecture 12a

Complex Fourier transforms

5 COMPLEX FOURIER TRASFORMS.

Recall the definition of the Fourier transform for an integrable function $f(x)$

$$\bar{f}(k) = \hat{\mathcal{F}}[f(x)] = \int_{-\infty}^{\infty} f(x) e^{ikx} \, dx$$

If $f(x)$ is continuous, then the inverse $f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{f}(k) e^{-ikx} \, dk$

Some important properties: $\hat{\mathcal{F}}[f'(x)] = -ik\bar{f}(k)$

$$\hat{\mathcal{F}}[x f(x)] = -i \frac{d\bar{f}}{dk}(k)$$

$$\hat{\mathcal{F}}[f * g] = \bar{f}(k)\,\bar{g}(k)$$ where $f*g = \int_{-\infty}^{\infty} f(s)\, g(x-s)\, ds$ is the convolution of f and g.

We extend the definition to non-integrable f by allowing k to take complex values.

Suppose $f(x) = O\left(e^{c|x|}\right)$ as $|x| \to \infty$, where c is a real constant.

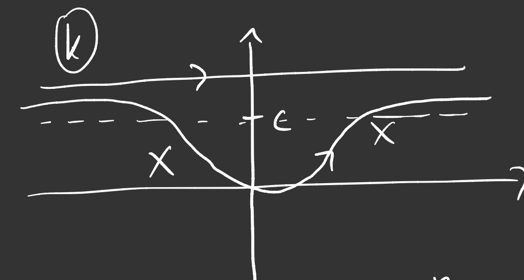
Define $f_+(x) = \begin{cases} f(x) & x \geq 0 \\ 0 & x < 0 \end{cases}$ & $f_-(x) = \begin{cases} 0 & x \geq 0 \\ f(x) & x < 0 \end{cases}$

Then define the half-range transform

$$\bar{f}_+(k) = \int_0^\infty f(x) e^{ikx} \, dx = \int_0^\infty f(x) e^{-\mathrm{Im}(k) x} e^{i\mathrm{Re}(k) x} \, dx \qquad \text{for } \mathrm{Im}(k) > c$$

$\bar{f}_+(k)$ is a holomorphic function of k as $\mathrm{Im}(k) > c$, and it can usually be analytically continued into $\mathrm{Im}(k) < c$, although there will typically be some singularities in $\mathrm{Im}(k) < c$.

The inverse is $f_+(x) = \frac{1}{2\pi} \int_{-\infty + i\alpha}^{\infty + i\alpha} \bar{f}_+(k) e^{-ikx} \, dk$



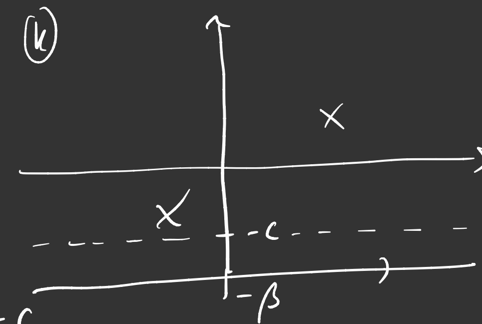
where $\alpha > c$.

The inversion contour can be deformed into $\mathrm{Im}(k) < c$, provided it stays above the singularities of $\bar{f}_+(k)$.

Similarly, we define $\bar{f}_-(k) = \int_{-\infty}^{0} f(x) e^{ikx} dx$ for $\mathrm{Im}(k) < -c$

The inverse is given by

$$f_-(x) = \frac{1}{2\pi} \int_{-\infty - i\beta}^{\infty - i\beta} \bar{f}_-(k) e^{-ikx} dk, \quad \text{with } -\beta < -c$$

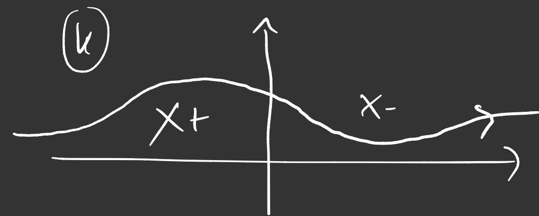


Again, the inversion contour can be deformed into $\mathrm{Im}(k) > -c$, provided it passes below the singularities of $\bar{f}_-(k)$.

If there is an overlap region, Ω say, where both $\bar{f}_+(k)$ and $\bar{f}_-(k)$ are holomorphic, then

$\bar{f}(k) = \bar{f}_+(k) + \bar{f}_-(k)$ is the Fourier transform, defined on Ω, with inverse

$$f(x) = \frac{1}{2\pi} \int_{\Gamma} \bar{f}(k) e^{-ikx} dk$$



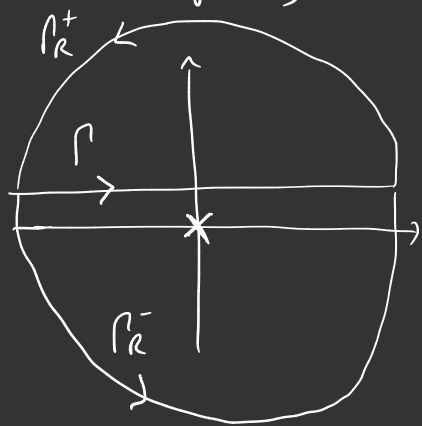
where Γ must pass above any singularities of $\bar{f}_+$ and below those of $\bar{f}_-$.

Lecture 12b

<u>Example</u> $f(x) = H(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}$

$$\bar{H}(k) = \int_0^\infty e^{ikx}\,dx = \frac{1}{ik} e^{ikx}\Big|_0^\infty = \frac{i}{k} \quad \text{for } \operatorname{Im}(k) > 0 \quad (c = 0 \text{ in the general theory})$$

$\bar{H}$ is originally defined for $\operatorname{Im}(k) > 0$, but is clearly holomorphic on $\mathbb{C} \setminus \{0\}$.



The inversion contour must pass above this singularity.

$$H(x) = \frac{1}{2\pi} \int_\Gamma \bar{H}(k) e^{-ikx}\,dk$$

For $x < 0$, $H(x) = \lim_{R \to \infty} -\frac{1}{2\pi} \int_{\Gamma_R^+} \bar{H}(k) e^{-ikx}\,dk = 0$

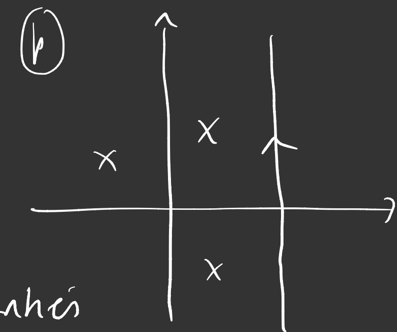
For $x > 0$, $H(x) = \lim_{R \to \infty} \frac{1}{2\pi} \left[\int_{\Gamma_R^-} \bar{H}(k) e^{-ikx}\,dk - 2\pi i \operatorname{res}\left(\bar{H}(k) e^{-ikx}, k=0\right) \right]$

$= 1$

The <u>Laplace transform</u> $\hat{f}(p)$ is obtained by setting $k = ip$ in the definition of $\bar{F}_+(k)$

i.e. $$\hat{f}(p) = \int_0^\infty f(x)\, e^{-px}\, dx$$

and the inverse is given by $$f(x) = \frac{1}{2\pi i} \int_{\alpha - i\infty}^{\alpha + i\infty} \hat{f}(p)\, e^{px}\, dp$$



where $\alpha > c$ such that the contour is to the right of all singularities of $\hat{f}(p)$.

Example use of the Fourier transform (real k)

Solve $\nabla^2 u = 0$ in $y > 0$, with $u(x,0) = u_0(x)$ & u bounded as $x^2+y^2 \to \infty$.

F.T. in x $\Rightarrow$ $-k^2\bar{u} + \dfrac{\partial^2 \bar{u}}{\partial y^2} = 0$ in $y > 0$, with $\bar{u}(k,0) = \bar{u}_0(k)$

& $\bar{u}$ bounded as $y \to \infty$.

Hence $\bar{u}(k,y) = A(k)e^{ky} + B(k)e^{-ky}$

But $\bar{u} \to 0$ as $y \to \infty$, so if $k > 0$, $A(k) = 0$ and if $k < 0$, $B(k) = 0$.

$B(k) = \bar{u}_0(k)$ $\qquad$ $A(k) = \bar{u}_0(k)$.

so $\bar{u}(k,y) = \bar{u}_0(k)e^{-|k|y}$

We note that $\mathcal{F}^{-1}\left[e^{-|k|y}\right] = \dfrac{y}{\pi(x^2+y^2)}$, so by the convolution theorem,

$$u(x,y) = \int_{-\infty}^{\infty} u_0(s) \frac{y}{\pi\left((x-s)^2+y^2\right)} \, ds$$