Lecture 13a

Applications of complex transforms

<u>Example</u> Solve $\nabla^2 u = 0$ in $y>0$, with u bounded as $x^2+y^2 \to \infty$, with $u(x,0) = H(x)$

[we can solve this easily in polar coordinates, $u = 1 - \frac{\theta}{\pi}$, but we'll use Fourier transforms]

Transform in x : $\bar{u}(k,y) = \int_{-\infty}^{\infty} u(x,y) e^{ikx} \, dx$.

$\Rightarrow \; -k^2 \bar{u} + \frac{\partial^2 \bar{u}}{\partial y^2} = 0$ for $y>0$, with $\bar{u}(k,y)$ bounded as $y \to \infty$

$\bar{u}(k,0) = \frac{i}{k}$ for $\operatorname{Im} k > 0$

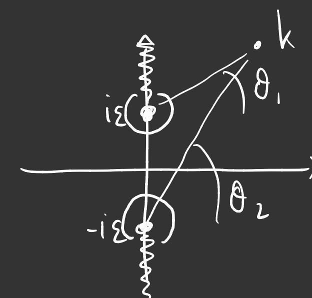
Solving $\Rightarrow$ $\bar{u}(k,y) = A(k) e^{ky} + B(k) e^{-ky}$

$= \frac{i}{k} e^{-|k| y}$ (so that $\bar{u} \to 0$ as $y \to \infty$)

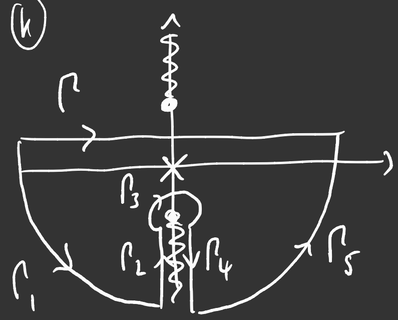
But $|k|$ is not holomorphic, so we approximate it by $|k| \approx (k^2+\varepsilon^2)^{1/2}$

we take $(k^2+\varepsilon^2)^{1/2} = |k^2+\varepsilon^2|^{1/2} e^{i(\theta_1+\theta_2)/2}$, with $\theta_1 \in \left(-\frac{3\pi}{2}, \frac{\pi}{2}\right]$, $\theta_2 \in \left(-\frac{\pi}{2}, \frac{3\pi}{2}\right]$

(with this definition $(k^2+\varepsilon^2)^{1/2}$ is real & positive on the real axis)



So we have $\bar{u}(k,y) = \frac{i}{k} e^{-y(k^2+\varepsilon^2)^{1/2}}$ & then $u(x,y) = \frac{1}{2\pi} \int_\Gamma \frac{i}{k} e^{-y(k^2+\varepsilon^2)^{1/2} - ikx} \, dk$



To calculate the integral, close in the Uhp for $x<0$, and the Lhp for $x>0$, diverting around the branch cuts.

eg. $x>0$, we have $\int_\Gamma = \underbrace{\int_{\Gamma_1}}_{\to 0} + \int_{\Gamma_2} + \underbrace{\int_{\Gamma_3}}_{\to 0} + \int_{\Gamma_4} + \underbrace{\int_{\Gamma_5}}_{\to 0} - \underbrace{2\pi i \operatorname{res}(k=0)}_{ie^{-y\varepsilon}}$

On Γ_2, write $k=-it$, $t \in (\varepsilon, \infty)$, $(k^2+\varepsilon^2)^{1/2} = i\sqrt{t^2-\varepsilon^2}$

Γ_4 $\qquad (k^2+\varepsilon^2)^{1/2} = -i\sqrt{t^2-\varepsilon^2}$

so $\int_{\Gamma_2} + \int_{\Gamma_4} = \int_\varepsilon^\infty \frac{i}{t} \underbrace{\left(-e^{-yi\sqrt{t^2-\varepsilon^2} - tx} + e^{+yi\sqrt{t^2-\varepsilon^2} - tx} \right)}_{2ie^{-tx} \sin\left(y\sqrt{t^2-\varepsilon^2}\right)} dt$

So, pulling together, $u(x,y) = \frac{1}{2\pi}\left[\int_\varepsilon^\infty \frac{-2\sin\left(y\sqrt{t^2-\varepsilon^2}\right)}{t} e^{-tx}\,dt - 2\pi i\left(i e^{-y\varepsilon}\right)\right]$

$$\longrightarrow 1 - \frac{1}{\pi}\underbrace{\int_0^\infty \frac{\sin(yt)}{t} e^{-tx}\,dt}_{\circledast\ \tan^{-1}\left(\frac{y}{x}\right)} = 1 - \frac{1}{\pi}\tan^{-1}\left(\frac{y}{x}\right)$$

as expected.

(*) This comes from considering another contour integral!

$$\int_\Gamma \frac{e^{-s}}{s}\,ds = 0$$

with $\phi = \tan^{-1}\left(\frac{y}{x}\right)$ (exercise)

Lecture 13 b

Integral solutions of differential equations

We can think of the Fourier transform as a method for finding integral solutions of the form

$$y(x) = \frac{1}{2\pi}\int_\Gamma \bar{y}(k)e^{-ikx}\,dk.$$

More generally, we can seek solutions to certain ODEs of the form $y(x) = \int_\Gamma g(s)e^{xs}\,ds$ for some function $g(s)$ and some contour Γ.

eg. $\frac{dy}{dx} = xy$. Substitute in the solution of the form above,

$$\int_\Gamma g(s)\,s\,e^{xs}\,ds = \int_\Gamma x\,g(s)e^{xs}\,ds$$

$$= \left[g(s)e^{xs}\right]_\Gamma - \int_\Gamma g'(s)e^{xs}\,ds.$$

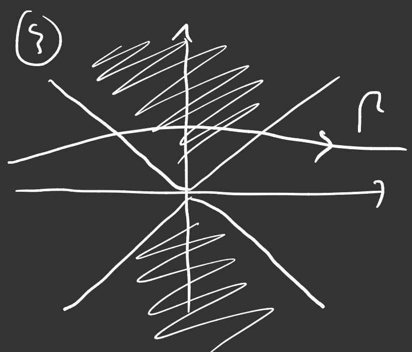
$$\Rightarrow \int_\Gamma \left[g'(s) + s\,g(s)\right]e^{xs}\,ds - \left[g(s)e^{xs}\right]_\Gamma = 0 \quad \forall x.$$

We can make sure this holds by choosing g to satisfy $g'(s) = -sg(s)$

and choosing Γ so that $\left[g(s)e^{xs}\right]_\Gamma = 0$ (either Γ closed, or $g(s)e^{xs} \to 0$ at end pts).

So we need $g' = -sg \Rightarrow g = Ce^{-s^2/2} \Rightarrow y(x) = C\int_\Gamma e^{-s^2/2 + xs}\,ds.$

To get a non-trivial solution we need Γ to go to infinity, but it must do so somewhere where $\mathrm{Re}(s^2/2) > 0$, so that $g(s)e^{xs} \to 0$ there. (i.e. the unshaded regions)



From the diagram, we can just take Γ along the real axis

so $$y(x) = C\int_{-\infty}^{\infty} e^{-\xi^2/2 + x\xi}\,d\xi = Ce^{x^2/2}\int_{-\infty}^{\infty} e^{-(\xi-x)^2/2}\,d\xi$$

$$= \hat{C}e^{x^2/2},$$ as expected.

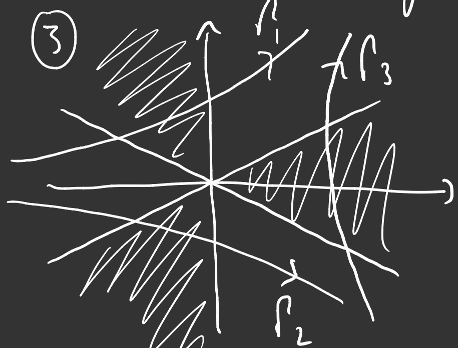
This method is more useful for higher-order equations, when coefficients are linear x.

eg. Airy's equation: $\frac{d^2y}{dx^2} + xy = 0$. Write soln as $y(x) = \int_\Gamma g(s) e^{xs}\,ds$

$$\Rightarrow 0 = \int_\Gamma [g(s)s^2 + xg(s)]e^{xs}\,ds = [g(s)e^{xs}]_\Gamma - \int_\Gamma [g'(s) - s^2 g(s)]e^{xs}\,ds$$

So we have $g' = s^2 g \Rightarrow g = Ce^{s^3/3} \Rightarrow \boxed{y(x) = C\int_\Gamma e^{s^3/3 + xs}\,ds.}$

We must choose Γ to go to ∞ somewhere where $\mathrm{Re}(s^3/3) < 0$



eg. Γ_1 or Γ_2 as in the diagram.

Lecture 14a

Wiener-Hopf method

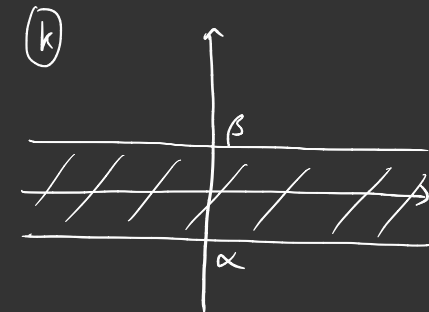
6 WIENER-HOPF METHOD

The essential idea of this method is as follows:

Suppose $\alpha < \beta$.

Suppose $f_+(k)$ is holomorphic in $\mathrm{Im}(k) > \alpha$, with $f_+(k) \to 0$ as $k \to \infty$

Suppose $g_-(k)$ is holomorphic in $\mathrm{Im}(k) < \beta$, with $g_-(k) \to 0$ as $k \to \infty$



& suppose $f_+(k) = g_-(k)$ on the region where they overlap.

By the identity theorem, $g_-(k)$ must be the analytic continuation of $f_+(k)$ into $\mathrm{Im}(k) < \alpha$ (and vice versa, f_+ is the continuation of g_- into $\mathrm{Im}(k) > \beta$), so together they define an entire function, $E(k)$, say.

But $E(k) \to 0$ as $k \to \infty$, so Liouville's theorem $\Rightarrow$ $E(k) = 0$. So $\underline{f_+(k) = g_-(k) = 0}$.

Example ODE problem.

$$\frac{d^2y}{dx^2} + b^2 y = 0 \qquad x<0 \qquad [y]_-^+ = 0, \ \left[\frac{dy}{dx}\right]_-^+ = 1 \ \text{ at } x=0.$$

$$\frac{d^2y}{dx^2} + a^2 y = 0 \qquad x>0 \qquad y \to 0 \text{ as } |x| \to \infty, \ \operatorname{Im} a>0, \operatorname{Im} b>0$$

$$a \neq b.$$

We assume (and later check) that $y = O(e^{\alpha x})$ as $x \to \infty$, and $y = O(e^{\beta x})$ as $x \to -\infty$, with $\alpha < \beta$.

Then take half-range Fourier transforms $\left(\bar{y}_-(k) = \int_{-\infty}^{0} y(x) e^{ikx} dx \right)$

$$\Rightarrow \quad (-k^2+b^2)\bar{y}_-(k) + \frac{dy}{dx}(0-) - iky(0-) = 0 \qquad \text{for } \operatorname{Im} k < \beta$$

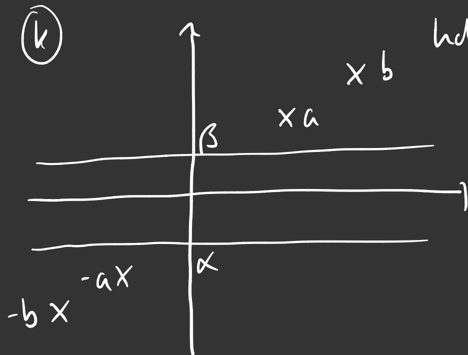
$$(-k^2+a^2)\bar{y}_+(k) - \frac{dy}{dx}(0+) + iky(0+) = 0 \qquad \text{for } \operatorname{Im} k > \alpha$$

Using the jump condition at $x=0$

$$\Rightarrow \quad \boxed{(k^2-a^2)\bar{y}_+(k) + (k^2-b^2)\bar{y}_-(k) = -1 \qquad \text{for } \alpha < \operatorname{Im} k < \beta}$$

Divide by $(k-a)(k+b)$ $\Rightarrow \left(\frac{k+a}{k+b}\right)\bar{y}_+(k) + \left(\frac{k-b}{k-a}\right)\bar{y}_-(k) = \frac{-1}{(k-a)(k+b)} = \frac{1}{a+b}\left(\frac{1}{k+b} - \frac{1}{k-a}\right)$

$$\Rightarrow \underbrace{\left(\frac{k+a}{k+b}\right)\bar{y}_+(k) - \frac{1}{(a+b)(k+b)}}_{\substack{\text{holomorphic in} \\ \operatorname{Im} k > \max(\alpha, -\operatorname{Im}(b))}} = \underbrace{-\left(\frac{k-b}{k-a}\right)\bar{y}_-(k) - \frac{1}{(a+b)(k-a)}}_{\substack{\text{holomorphic in} \\ \operatorname{Im} k < \min(\beta, \operatorname{Im}(a))}} \quad \text{for } \alpha < \operatorname{Im} k < \beta$$



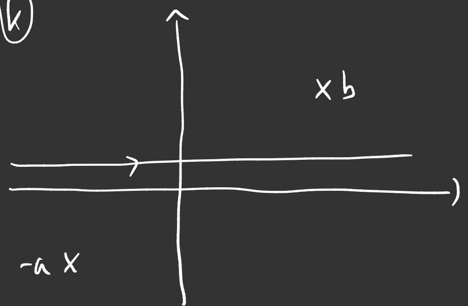
The LHS and RHS are equal on the overlapping strip, so define the analytic continuation of each other to the whole complex plane.
Together they define an entire function, $E(k)$, say.
But, since LHS & RHS $\to 0$ as $k \to \infty$ (since $\bar{y}_+$ & $\bar{y}_- \to 0$ as $k \to \infty$)
$E(k) \to 0$ as $k \to \infty$, so Liouville $\Rightarrow E(k) = 0$.

$$\Rightarrow \text{LHS} = \text{RHS} = 0$$

Lecture 14b

We have found that $\bar{y}_+(k) = \dfrac{1}{(a+b)(k+a)}$ & $\bar{y}_-(k) = \dfrac{-1}{(a+b)(k-b)}$

(k)



We close the inversion contour in the lhp & uhp respectively.

So $x>0$, $y(x) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} \bar{y}_+(k) e^{-ikx}\, dk = -\dfrac{1}{2\pi} 2\pi i \left(\dfrac{1}{a+b} e^{iax} \right)$

$$= -\frac{i}{a+b} e^{iax}.$$

$x<0$, $y(x) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} \bar{y}_-(k) e^{-ikx}\, dk = \dfrac{1}{2\pi} 2\pi i \left(-\dfrac{1}{a+b} e^{-ibx} \right)$

$$= -\frac{i}{a+b} e^{-ibx}$$

We assumed that $y = O(e^{\alpha x})$ as $x \to \infty$, and we now see that $\alpha = -\mathrm{Im}\, a$, and similarly we see that $\beta = \mathrm{Im}\, b$, so indeed $\alpha < \beta$, as assumed.

Wiener-Hopf decomposition

The key step in this process is to take an expression of the form

$$F(z)\,\bar{u}_+(z) + \bar{v}_-(z) = G(z) \qquad \text{for } \alpha < \mathrm{Im}\, z < \beta$$

where F, G are given, and $\bar{u}_+, \bar{v}_-$ are the unknowns, and rewrite it as

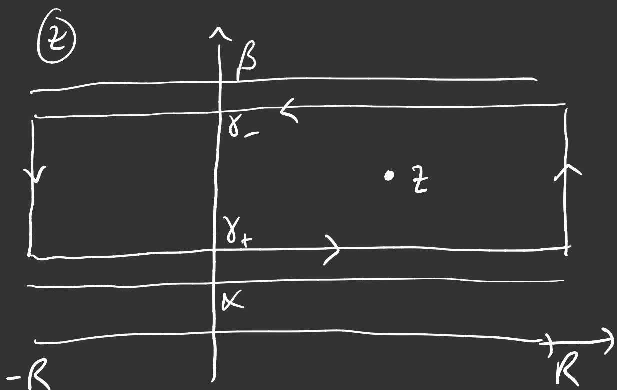
$f_+(z) = g_-(z)$ for some f_+, g_-, which we can conclude are zero using their behaviour at ∞.

This may require writing $F(z) = \dfrac{F_+(z)}{F_-(z)}$, and $G(z) = G_+(z) - G_-(z)$

where '+' means holomorphic in $\mathrm{Im}\,\underset{z}{\cancel{k}} > \alpha$, and '−' means holomorphic in $\mathrm{Im}\,\underset{z}{\cancel{k}} < \beta$.

This 'splitting' can often be done by inspection, or using partial fractions, but there is also a general method for doing it.

Sum decomposition. Let $G(z)$ be holomorphic on $\alpha < \operatorname{Im} z < \beta$, with $G(z) \to 0$ as $z \to \infty$.



By Cauchy's integral formula, $G(z) = \dfrac{1}{2\pi i} \displaystyle\int_\Gamma \dfrac{G(s)}{s-z}\, ds$

Taking Γ to be the rectangle shown, and letting $R \to \infty$, so the contributions from the ends vanish, we have

$$G(z) = \underbrace{\frac{1}{2\pi i}\int_{-\infty+\gamma_+ i}^{\infty+\gamma_+ i} \frac{G(s)}{s-z}\, ds}_{G_+(z)} - \underbrace{\frac{1}{2\pi i}\int_{-\infty+\gamma_- i}^{\infty+\gamma_- i} \frac{G(s)}{s-z}\, ds}_{G_-(z)}$$

$G_\pm$ defined in this way are holomorphic on $\operatorname{Im} z > \gamma_+$ and $\operatorname{Im} z < \gamma_-$ respectively, so this gives the required decomposition.

Ratio decomposition Let $F(z)$ be holomorphic in $\alpha < \operatorname{Im} z < \beta$, with $F(z) \to 1$ as $z \to \infty$.

We want to write $F(z) = \dfrac{F_+(z)}{F_-(z)}$, so take logs $\Rightarrow$ $\underbrace{\log F}_{G} = \underbrace{\log F_+}_{G_+} - \underbrace{\log F_-}_{G_-}$

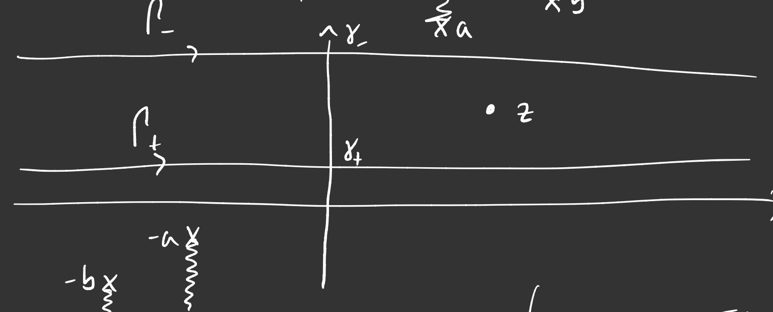
This is of the form of a sum decomposition, so we can apply the same procedure to $\log F$,

giving
$$F_\pm(z) = \exp\left[\frac{1}{2\pi i}\int_{-\infty+\gamma_\pm i}^{\infty+\gamma_\pm i} \frac{\log F(s)}{s-z}\, ds\right]$$

Lecture 15a

Application of Wiener-Hopf method to an integral equation.

Recall that we can write $F(z) = \dfrac{F_+(z)}{F_-(z)}$ where $F_\pm(z) = \exp\left[\dfrac{1}{2\pi i}\displaystyle\int_{\Gamma_\pm} \dfrac{\log F(s)}{s-z}\, ds\right]$



eg. $F(z) = \dfrac{z^2-a^2}{z^2-b^2}$ where $\operatorname{Im} a > 0$, $\operatorname{Im} b > 0$ $\quad$ (note that $F(z) = \underbrace{\dfrac{z+a}{z+b}}_{F_+} \Big/ \underbrace{\dfrac{z-b}{z-a}}_{F_-}$)

Using the formula,

$$F_+(z) = \exp\left[\underbrace{\frac{1}{2\pi i}\int_{\Gamma_+}\frac{\log(s+a)-\log(s+b)}{s-z}\,ds}_{\text{close in Uhp, } = \text{res}(s=z) = \log(z+a)-\log(z+b)} + \underbrace{\frac{1}{2\pi i}\int_{\Gamma_+}\frac{\log(s-a)-\log(s-b)}{s-z}\,ds}_{\text{close in Lhp, } = 0}\right] = \frac{z+a}{z+b}$$

Similarly, $F_-(z) = \dfrac{z-b}{z-a}$

Example application to an integral equation

Find $f(x)$, smooth & bounded, such that $\int_0^\infty f(t)\, e^{-|x-t|}\, dt = f(x)$ for $x \geq 0$

Define $f_+(x) = \begin{cases} 0 & x<0 \\ f(x) & x \geq 0 \end{cases}$ & $h_-(x) = \begin{cases} \int_0^\infty f(t)\, e^{-|x-t|}\, dt & x<0 \\ 0 & x \geq 0 \end{cases}$ & $K(x) = e^{-|x|}$

Then the equation is $\int_{-\infty}^{\infty} f_+(t)\, K(x-t)\, dt = f_+(x) + h_-(x)$ for $-\infty < x < \infty$.

Note that since $f_+(x)$ is bounded, $\bar{f}_+(k)$ is holomorphic for $\operatorname{Im} k > 0$. Also $h_-(x) = e^{x} \int_0^\infty f(t)\, e^{-t}\, dt$ for $x<0$, so $\bar{h}_-(k)$ is holomorphic for $\operatorname{Im} k < 1$. Also $K(x)$ has $\bar{K}(k) = \frac{2}{1+k^2}$ for $-1 < \operatorname{Im} k < 1$

So taking the Fourier transform $\Rightarrow$ $\bar{f}_+(k)\, \bar{K}(k) = \bar{f}_+(k) + \bar{h}_-(k)$ for $0 < \operatorname{Im} k < 1$

$$\Rightarrow \boxed{\left(\frac{1-k^2}{1+k^2}\right) \bar{f}_+(k) = \bar{h}_-(k) \quad \text{for } 0 < \operatorname{Im} k < 1}$$

Aside: since $f(x)$ is smooth & bounded, we can expect $\bar{f}_+(k) = O(1/k)$ as $k \to \infty$.

Why? $$\bar{f}_+(k) = \int_0^\infty f(x) e^{ikx}\,dx = \underbrace{\frac{1}{ik} f(x) e^{ikx} \Big|_0^\infty}_{-\frac{f(0)}{ik}} - \frac{1}{ik} \underbrace{\int_0^\infty f'(x) e^{ikx}\,dx}_{\to 0 \text{ by Riemann-Lebesgue lemma as } k\to\infty}$$

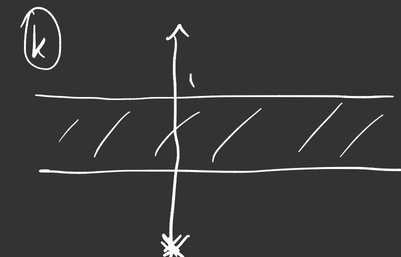
$$= \frac{if(0)}{k} + o\left(\frac{1}{k}\right)$$

Similarly, since $h_-(x)$ is bounded, we can expect $\bar{h}_-(k) = O\left(\frac{1}{k}\right)$ as $k \to \infty$.

Lecture 15b

$$\boxed{\left(\frac{1-k^2}{1+k^2}\right)\bar{f}_+(k) = \bar{h}_-(k) \qquad \text{for } 0<\operatorname{Im} k<1}$$

Note that $\frac{1-k^2}{1+k^2} = \underbrace{\frac{1-k^2}{k+i}}_{F_+} \Big/ \underbrace{k-i}_{F_-}$ so we should multiply by $(k-i)$ to find

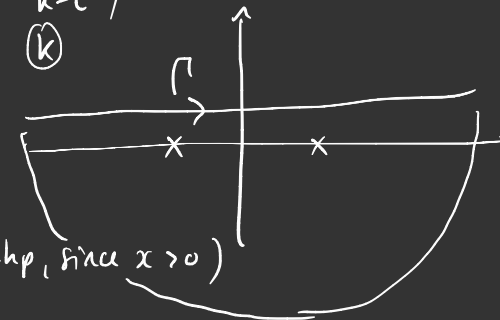


$$\left(\frac{1-k^2}{k+i}\right)\bar{f}_+(k) = (k-i)\bar{h}_-(k) \quad \text{for } 0<\operatorname{Im} k<1$$

The LHS & RHS are holomorphic in $\operatorname{Im} k > 0$ & $\operatorname{Im} k<1$ respectively, so together they define an entire function, $E(k)$.

Since $\bar{f}_+(k)$ and $\bar{h}_-(k)$ are $O(1/k)$ as $k\to\infty$, $E(k) = O(1)$ as $k\to\infty$ (i.e. bounded), so Liouville's theorem $\Rightarrow$ $E(k) = C$, constant.

So we have $\bar{f}_+(k) = \dfrac{C(k+i)}{1-k^2}$ (& $\bar{h}_-(k) = \dfrac{C}{k-i}$)



So $f_+(x) = \dfrac{C}{2\pi} \displaystyle\int_\Gamma \underbrace{\frac{k+i}{1-k^2} e^{-ikx}}_{-\frac{(k+i)}{(k-1)(k+1)} e^{-ikx}} \, dk$

(close in the lhp, since $x > 0$)

$$= -\frac{C}{2\pi} 2\pi i \left[\text{res}(k=1) + \text{res}(k=-1) \right] = -Ci \left[\underbrace{-\frac{1+i}{2} e^{-ix} + \frac{-1+i}{2} e^{ix}}_{-\cos x - \sin x} \right]$$

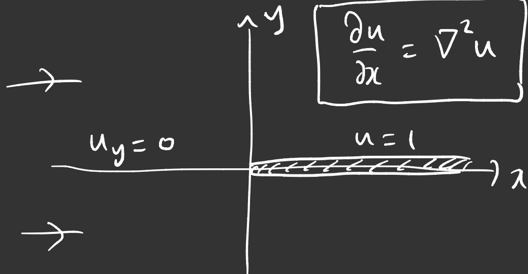
$$= \hat{C} (\cos x + \sin x)$$

Lecture 16 a

Application of Wiener-Hopf method to a mixed boundary value problem

Find the steady state temperature $u(x,y)$ of fluid flow past a heated plate.

$$\boxed{\frac{\partial u}{\partial x} = \nabla^2 u}$$



Consider $y > 0$.

We have $\boxed{u \to 0 \text{ as } y \to \infty}$

& $u(x,0) = 1$ for $x \geq 0$

$u_y(x,0) = 0$ for $x < 0$

It is of particular interest what is the heat flux out of the plate, ie. $u_y(x,0)$ $x \geq 0$.

To deal with the mixed boundary conditions, we define

$$f_-(x) = \begin{cases} u(x,0) & x < 0 \\ 0 & x \geq 0 \end{cases} \quad \& \quad g_+(x) = \begin{cases} 0 & x < 0 \\ u_y(x,0) & x \geq 0 \end{cases}$$

(ie the unknown values of u/u_y on the boundary)

Then we have $\boxed{u(x,0) = f_-(x) + H(x), \quad u_y(x,0) = g_+(x)}$ for all x.

Assume $f_-(x) = O(e^{\beta x})$ as $x \to -\infty$, and $g_+(x) = O(e^{\alpha x})$ as $x \to \infty$, with $\alpha < \beta$

Transform in x, so $\bar{u}(k,y)$ satisfies

$$-ik\bar{u} = -k^2\bar{u} + \frac{\partial^2\bar{u}}{\partial y^2} = 0 \quad \text{for } y>0 \quad \Rightarrow \quad \frac{\partial^2\bar{u}}{\partial y^2} = (k^2-ik)\bar{u} \quad \text{for } y>0$$

with $\bar{u}(k,y) \to 0$ as $y \to \infty$

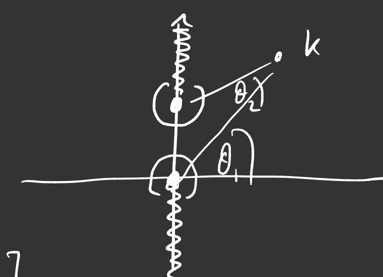
& $\bar{u}(k,0) = \bar{f}_-(k) + \frac{i}{k}$ for $0 < \text{Im}\, k < \beta$

$\frac{\partial\bar{u}}{\partial y}(k,0) = \bar{g}_+(k)$ for $\text{Im}\, k > \alpha$.

Solving the equation gives $\boxed{\bar{u}(k,y) = A(k)e^{-(k^2-ik)^{1/2}y}}$

where $(k^2-ik)^{1/2} = k^{1/2}(k-i)^{1/2} = |k|^{1/2}|k-i|^{1/2}e^{i(\theta_1+\theta_2)/2}$ with $\theta_1 \in (-\frac{\pi}{2}, \frac{3\pi}{2}]$

$\theta_2 \in (-\frac{3\pi}{2}, \frac{\pi}{2}]$

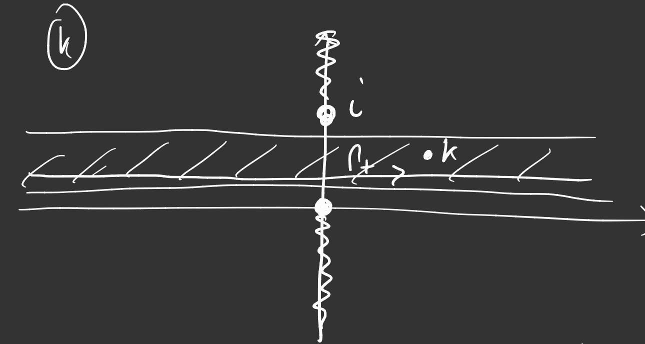


Applying conditions at $y=0$ $\Rightarrow$ $A(k) = \bar{f}_-(k) + \frac{i}{k}$ & $-(k^2-ik)^{1/2}A(k) = \bar{g}_+(k)$

Eliminate $A(k)$ $\Rightarrow$ $\boxed{\bar{g}_+(k) + (k^2-ik)^{1/2}\bar{f}_-(k) = -\frac{i(k^2-ik)^{1/2}}{k}}$ for $\alpha < \text{Im}\, k < \beta$ (assume $\alpha \geq 0$)

Lecture 16b

$$\boxed{\bar g_+(k) + (k^2 - ik)^{1/2}\,\bar f_-(k) = -\frac{i(k^2-ik)^{1/2}}{k}}$$



Divide by $k^{1/2}$ (a '+' function)

$$\Rightarrow \quad \frac{\bar g_+(k)}{k^{1/2}} + (k-i)^{1/2}\,\bar f_-(k) = \underbrace{-\frac{i(k-i)^{1/2}}{k}}$$

$$G(k) = \underbrace{-\frac{i(-i)^{1/2}}{k}}_{G_+(k)} - \underbrace{\left(\frac{i(k-i)^{1/2}}{k} - \frac{i(-i)^{1/2}}{k}\right)}_{G_-(k)}$$

$(-i)^{1/2} = e^{-i\pi/4}$ given our defn of $(k-i)^{1/2}$.

Aside: using the formula

$$G_+(k) = \frac{1}{2\pi i}\int_{\Gamma_+}\frac{G(s)}{s-k}\,ds = \frac{1}{2\pi i}\int_{\Gamma_+}\frac{-i(s-i)^{1/2}}{s(s-k)}\,ds = -\mathrm{res}\left(\frac{-i(s-i)^{1/2}}{s(s-k)}, s=0\right) = -\frac{i(-i)^{1/2}}{k}.$$

Then $G_-(k) = G_+(k) - G(k)$.

So we have $\dfrac{\bar{g}_+(k)}{k^{1/2}} + \dfrac{ie^{-i\pi/4}}{k} = -(k-i)^{1/2}\bar{f}_-(k) - \dfrac{i(k-i)^{1/2}}{k} + \dfrac{ie^{-i\pi/4}}{k}$ for $\alpha < \operatorname{Im} k < \beta$

Since LHS is holomorphic on $\operatorname{Im} k > \alpha$, and RHS is holomorphic on $\operatorname{Im} k < \beta$, they are the analytic continuation of each other, so together define an entire function, $E(k)$.

We expect, since u is continuous, that $\bar{f}_-(k) = o\left(\frac{1}{k}\right)$ as $k \to \infty$, and $\bar{g}_+(k) = O\left(\frac{1}{k^{1/2}}\right)$ as $k \to \infty$.

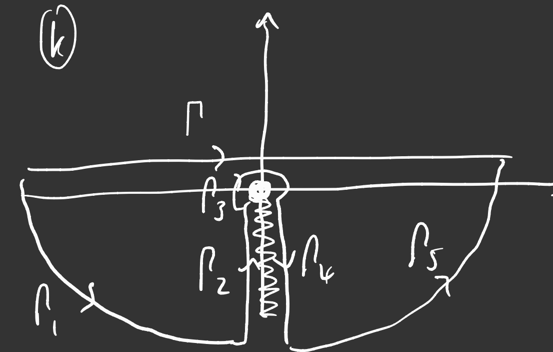
(since anticipate $u_y(x,0)$ may have a square-root singularity)

Hence, we must have $E(k) \to 0$ as $k \to \infty$. So Liouville's theorem $\Rightarrow E(k) = 0$.

$$\Rightarrow \bar{g}_+(k) = -\frac{e^{i\pi/4}}{k^{1/2}} \quad \& \quad \bar{f}_-(k) = -\frac{i}{k} + \frac{e^{i\pi/4}}{k(k-i)^{1/2}}$$

$$\Rightarrow \boxed{\bar{u}(k,y) = \frac{e^{i\pi/4}}{k(k-i)^{1/2}} e^{-(k^2-ik)^{1/2} y}}$$

For the heat flux $\frac{\partial u}{\partial y}(x,0)$, $x>0$, we need $g_+(x)$



$$g_+(x) = \frac{1}{2\pi}\int_\Gamma -\frac{e^{i\pi/4}}{k^{1/2}}\, e^{-ikx}\, dk.$$

Close in the LHP (since $x>0$), & note that

$\int_{\Gamma_1}, \int_{\Gamma_5}, \int_{\Gamma_3} \to 0$ in the appropriate limits, so $\int_\Gamma = 2\int_{\Gamma_4} \leftarrow$ $k=-it$, $k^{1/2} = t^{1/2}e^{-i\pi/4}$

$$\text{Hence } g_+(x) = -\frac{2}{2\pi}\int_0^\infty \frac{e^{i\pi/4}}{e^{-i\pi/4}t^{1/2}}\, e^{-tx}\,(-i\,dt) = -\frac{1}{\pi}\underbrace{\int_0^\infty t^{-1/2}e^{-tx}\,dt}$$

$tx = s^2 \quad dt = \frac{2s}{x}ds$

$$\frac{2}{x^{1/2}}\int_0^\infty e^{-s^2}ds = \sqrt{\frac{\pi}{x}}$$

$$= -\frac{1}{\sqrt{\pi x}}$$