Sheet 1: Review, conformal mapping

Q1 Use contour integration to evaluate

$$\int_{-1}^{1} \frac{\sqrt{1-x^2}}{x^2+1}\,\mathrm{d}x.$$

[*Hint: Consider the integral of* $\left(z^2-1\right)^{1/2} / \left(z^2+1\right)$ *with branch cut from* $z=-1$ *to* $z=1$*, around a large circle.*]

Q2 (a) Show that, if $\zeta^2 = z^2+1$ and $z = \mathrm{i} + r_1 \mathrm{e}^{\mathrm{i}\theta_1} = -\mathrm{i} + r_2 \mathrm{e}^{\mathrm{i}\theta_2}$, where $r_1,\ r_2 > 0$, $\theta_1,\ \theta_2 \in \mathbb{R}$, then

$$\zeta = \pm (r_1 r_2)^{1/2} \mathrm{e}^{\mathrm{i}(\theta_1+\theta_2)/2}.$$

Explain briefly why $z = \pm\mathrm{i}$ are the branch points of the multifunction $(z^2+1)^{1/2}$.

(b) Consider the branch defined by $f(z) = (r_1 r_2)^{1/2}\mathrm{e}^{\mathrm{i}(\theta_1+\theta_2)/2}$, with $-\pi/2 < \theta_1,\ \theta_2 \leq 3\pi/2$. State the values of $(\theta_1+\theta_2)/2$ on either side of the imaginary axis, and hence compute $f(\pm 0 + \mathrm{i}y)$ in terms of y for $y \in \mathbb{R}$. Deduce that the branch cut is $S = \{x + \mathrm{i}y : x = 0,\ |y| \leq 1\}$. Show that $f(z) \sim z$ as $|z| \to \infty$ (*i.e.* $f(z)/z \to 1$ as $|z| \to \infty$). Sketch the image of the cut z-plane $\mathbb{C}\backslash S$ under the map $\zeta = f(z)$.

(c) Consider instead the branch $f(z) = (r_1 r_2)^{1/2}\mathrm{e}^{\mathrm{i}(\theta_1+\theta_2)/2}$, with $-3\pi/2 < \theta_1 \leq \pi/2$ and $-\pi/2 < \theta_2 \leq 3\pi/2$. State the values of $(\theta_1+\theta_2)/2$ on either side of the imaginary axis, and hence compute $f(\pm 0 + \mathrm{i}y)$ in terms of y for $y \in \mathbb{R}$. Deduce that the branch cut is along the imaginary axis from $z = -\mathrm{i}\infty$ to $z = -\mathrm{i}$ and from $z = \mathrm{i}$ to $z = \mathrm{i}\infty$. Compute $f(x)$ in terms of x for $x \in \mathbb{R}$. Show that

$$f(z) \sim \begin{cases} z & \text{as } |z| \to \infty \text{ with } \mathrm{Re}(z) > 0, \\ -z & \text{as } |z| \to \infty \text{ with } \mathrm{Re}(z) < 0. \end{cases}$$

Sketch the images of the half-planes $\mathrm{Re}(z) > 0$ and $\mathrm{Re}(z) < 0$ under the map $\zeta = f(z)$.

Q3 (a) Find the image of the common part of the discs $|z-1| < 1$ and $|z+\mathrm{i}| < 1$ under the mapping $\zeta = 1/z$.

(b) Find the image of the strip $-\frac{\pi}{2} < x < \frac{\pi}{2}$ under the map $\zeta = \sin z$.

(c) Find the image of the strip $-\pi < y < \pi$ under the map $\zeta = z + \mathrm{e}^z$.

[*Hint: Where are the critical points of the map?*]

Q4 (a) D is the region exterior to the two circles $|z-1| = 1$ and $|z+1| = 1$. Find a conformal mapping from D to the exterior of the unit circle.

[*Hint: First apply an inversion with respect to the origin* ($z \mapsto 1/z$)*, then rotate and scale so as to use the exponential function to get to a half plane, from where use Möbius.*]

(b) Find a conformal map from the unit disc $|z| < 1$ onto the strip $-\frac{\pi}{2} < \eta < \frac{\pi}{2}$, taking the origin to the origin, 1 to $\xi = +\infty$, -1 to $\xi = -\infty$. (Thus, the upper half of the unit circle maps to $\eta = \frac{\pi}{2}$ and the lower half to $\eta = -\frac{\pi}{2}$.)

(c) Find a conformal map of the quarter-disc $0 < |z| < 1$, $0 < \arg z < \frac{\pi}{2}$ to the upper half-plane $\eta > 0$, taking 0 to 0, 1 to 1, and i to ∞.

Q5 The domain D consists of the right-hand half plane $x > 0$ with the circle $|z-a| = b$, $0 < b < a$, and its interior removed. Find the temperature $u(x,y)$ in steady heat flow if $u = 0$ on the y axis, $u = 1$ on $|z-a| = b$, and $u \to 0$ at infinity.

[*Hint: Show that the mapping* $\zeta = (z-\alpha)/(z+\alpha)$*, with* α *real and positive, takes* D *onto an annular region with the imaginary axis mapping to* $|\zeta| = 1$ *and show that, if* $\alpha^2 = a^2 - b^2$*, then the image of* D *is a concentric circular annulus.*]

Q6 Write down, as an integral, the Schwarz–Christoffel map from a half-plane to a rectangle, with the vertices being the images of the points $z = \pm 1$ and $z = \pm a$, where $a > 1$ is real. Explain why a cannot be specified arbitrarily, but is determined by the aspect ratio of the rectangle.