Sheet 2: Fluid flow

Q1 (a) Carefully define a branch of the function $\cosh^{-1}(Z)$ that is holomorphic in the upper half-plane. What is $\cosh^{-1}(0)$? What is the derivative of $\cosh^{-1}(Z)$?

(b) Show that the Schwarz–Christoffel map from the upper half-plane to the exterior of the half-strip $0 < x < \infty$, $-1 < y < 1$ has the form

$$z = A + C\left(Z\sqrt{Z^2-1} - \cosh^{-1} Z\right),$$

and find the constants A and C.

[*Hint: Map* $Z = \pm 1$ *to the finite corners of the domain, and* $Z = \infty$ *to the vertex at* $x = \infty$.]

(c) Hence find the complex potential $w(z)$ for potential flow past this obstacle with a uniform stream $(U_1, 0)$ at infinity.

[*Hint: Bearing in mind the behaviour of the mapping at infinity, think carefully about the potential in the* Z *plane: it is not 'constant* $\times$ Z.']

Q2 Inviscid irrotational fluid flows steadily in the domain Ω shown in figure 1, between a rigid wall ABC consisting of two semi-infinite straight line segments meeting at right angles, and a free surface $A'C'$.

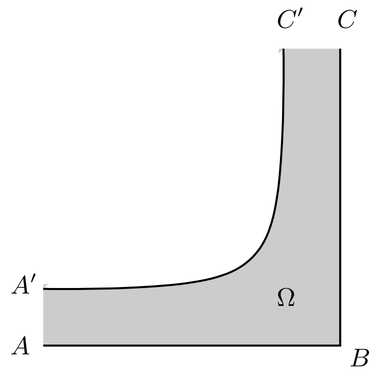

Figure 1: A jet climbing a wall.

The fluid layer has thickness 1 and velocity $(1, 0)$ far upstream, at AA'. The boundary value problem for the complex potential $w(z) = \phi + i\psi$ is that $w(z)$ is holomorphic in Ω, with

$$\psi = 0 \text{ on } ABC, \qquad \psi = 1, \quad |w'| = 1 \text{ on } A'C',$$

where $w'(z) = u - iv$ is the complex velocity. In addition, take the reference point for ϕ so that $w = 0$ at B.

(a) Show that flow domain in the potential plane (w) is a strip, while in the hodograph plane (w') it is a quarter of the unit circle.

(b) Show that the map to a half plane is

$$\zeta = e^{\pi w} = \left(\frac{(w')^2 - 1}{(w')^2 + 1}\right)^2.$$

(c) Parametrise the free surface $A'C'$ by $w' = e^{-i\theta}$, where $0 \le \theta \le \pi/2$. Show that

$$\zeta = -\tan^2\theta, \qquad \frac{dz}{d\theta} = \frac{1}{w'}\frac{dw}{d\zeta}\frac{d\zeta}{d\theta} = \frac{2}{\pi}(\operatorname{cosec}\theta + i\sec\theta) \qquad \text{on} \quad A'C'.$$

(d) Find parametric equations for the free surface from the real and imaginary parts of $\mathrm{d}z/\mathrm{d}\theta$. Check that it looks as it should.

Q3 A two-dimensional jet of inviscid irrotational fluid, of thickness $2h_\infty$ and moving to the right with speed 1, enters a semi-infinite rectangular cavity with walls at $y = \pm 1$ and $x = L$, as shown in figure 2; the y axis is tangent to the free surface.

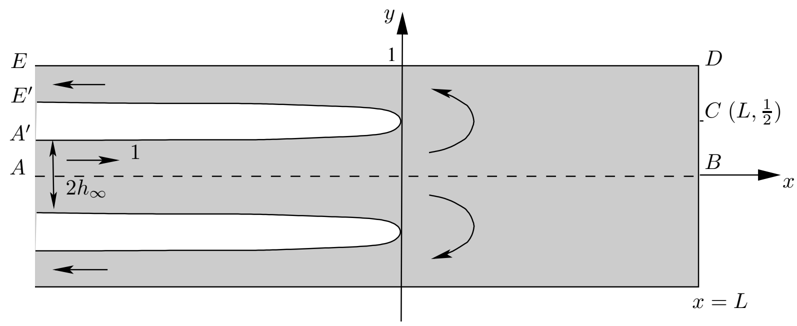

Figure 2: A jet entering a box.

The boundary value problem for the complex potential $w(z) = \phi + \mathrm{i}\psi$ for the upper half of the flow (within the strip $0 < y < 1$, $-\infty < x < L$) is that $w(z)$ is holomorphic in the fluid region, with

$$\psi = 0 \text{ on } ABCDE, \qquad \psi = h_\infty, \quad |w'| = 1 \text{ on } A'E',$$

where $w'(z) = u - \mathrm{i}v$ is the complex velocity. In addition, take the reference point for ϕ so that $w = 0$ at C.

(a) Sketch the flow domain in the potential and hodograph planes.

(b) Now consider the case $L = \infty$, with stagnant fluid far inside the cavity.

(i) Show that B, C and D coincide at the origin in the hodograph plane, so that the flow domain is the whole interior of the semicircle in the lower half plane.

(ii) Show that

$$\frac{\mathrm{d}w}{\mathrm{d}z} = \frac{1 - \mathrm{e}^{\pi w/2h_\infty}}{1 + \mathrm{e}^{\pi w/2h_\infty}} = -\tanh \frac{\pi w}{4h_\infty}.$$

Find w satisfying $w = \mathrm{i}h_\infty$ at $z = \mathrm{i}/2$, the tip of the air finger shown in figure 2.

(iii) Show that the free surface for this flow, $w = \phi + \mathrm{i}h_\infty$, $-\infty < \phi < \infty$, satisfies

$$\mathrm{e}^{-\pi x/2h_\infty} \cos\left(\frac{\pi(y - \frac{1}{2})}{2h_\infty}\right) = 1,$$

and show that $y \to \pm h_\infty$ as $x \to -\infty$ is only consistent if h_∞ takes a particular value, which you should find.

Q4 (a) The *harmonic moments* of a domain $D(t)$ are defined by

$$M_n(t) = \iint_{D(t)} z^n \,\mathrm{d}x\mathrm{d}y, \qquad n = 0, 1, 2, \ldots,$$

where $z = x + \mathrm{i}y$. What are the physical significance of $M_0(t)$ and $M_1(t)$?

If the boundary $\partial D(t)$ has outward normal velocity V_n, show (e.g. using Reynolds' Transport Theorem; see Part A Fluids & Waves) that

$$\frac{\mathrm{d}M_n}{\mathrm{d}t} = \oint_{\partial D} z^n V_n \,\mathrm{d}s.$$

(b) Use Green's Theorem on a region $R \subset \mathbb{R}^2$ to show that

$$\iint_R \frac{\partial G}{\partial \overline{z}}(z, \overline{z})\,\mathrm{d}x\,\mathrm{d}y = \frac{1}{2\mathrm{i}} \oint_{\partial R} G(z, \overline{z})\,\mathrm{d}z$$

for any sufficiently smooth function $G(z, \overline{z})$. Deduce that

$$M_n(t) = \frac{1}{2\mathrm{i}} \oint_{\partial D} z^n \overline{z}\,\mathrm{d}z.$$

(c) The domain $D(t)$ is a saturated region of a porous medium, in which flow is driven by a point source of strength Q at $z = 0$. The potential $\phi(x, y, t)$ satisfies Laplace's equation in $D(t)$, with $\phi \sim (Q/2\pi)\log r$ at the origin (where $r^2 = x^2 + y^2$), together with $\phi = 0$, $\partial\phi/\partial n = V_n$ on $\partial D(t)$.

Use Green's Second Identity on $D(t)$ with a small circle around $z = 0$ removed to show that

$$\frac{\mathrm{d}M_0}{\mathrm{d}t} = Q, \qquad \frac{\mathrm{d}M_n}{\mathrm{d}t} = 0, \quad n > 0.$$

(d) The map $z = F(\zeta, t)$ maps the unit disc $|\zeta| < 1$ onto $D(t)$, with $F(0, t) = 0$. Show that

$$M_n(t) = \frac{1}{2\mathrm{i}} \oint_{|\zeta|=1} F(\zeta, t)^n \overline{F(\zeta, t)} \frac{\partial F}{\partial \zeta}\,\mathrm{d}\zeta.$$

Now suppose that $F(\zeta, t)$ is a polynomial of degree m, with coefficients $a_j(t)$, $j = 1 \ldots m$. Making use of the fact that $\overline{\zeta} = 1/\zeta$ on $|\zeta| = 1$, show that $M_n(t) = 0$ for $n \geq m$.

Hence find the nonzero moments for the quadratic map $F(\zeta, t) = a_1(t)\zeta + a_2(t)\zeta^2$, and crosscheck with the solution of the differential equations given in lectures.

Find formulae for M_0 and M_{m-1} for a general polynomial of degree m with complex coefficients.

[*Green's Theorem states that for any (suitably smooth) scalar functions $P(x, y)$ and $Q(x, y)$ and region $R \subset \mathbb{R}^2$ with (suitably smooth) boundary ∂R,*

$$\iint_R \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) \mathrm{d}x\mathrm{d}y = \oint_{\partial R} (P\,\mathrm{d}y - Q\,\mathrm{d}x)\,.$$

A corollary is Green's Second Identity:

$$\iint_R \left(u\nabla^2 v - v\nabla^2 u\right) \mathrm{d}x\mathrm{d}y = \oint_{\partial R} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) \mathrm{d}s,$$

again for suitably smooth $u(x, y)$ and $v(x, y)$.]