C5.2 Elasticity and Plasticity

Lecture 8 — Nonlinear beam theory

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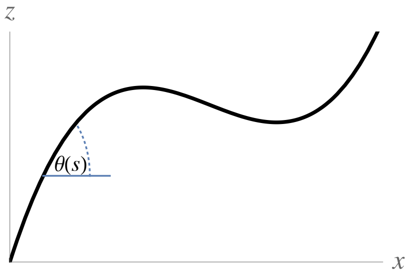
Nonlinear beam theory

- Now drop assumption of small transverse displacement.
- For simplicity neglect body force and assume steady state.

Describe shape of beam by angle $\theta(s)$ where s is arc-length.



If we solve for $\theta(s)$, can recover shape of beam using...

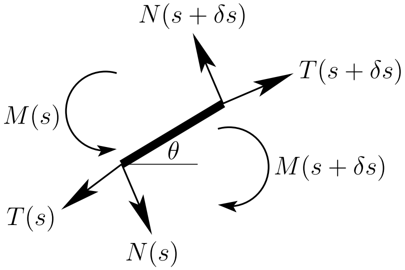
$$\boxed{\frac{\mathrm{d}x}{\mathrm{d}s} = \cos\theta} \qquad \boxed{\frac{\mathrm{d}z}{\mathrm{d}s} = \sin\theta}$$

Nonlinear beam theory

- Again consider a small segment $[s, s+\delta s]$:

Now define shear force N in normal direction.

(when $\theta = O(1)$ it makes a difference...)



- Force and moment balances give...

$$\frac{\mathrm{d}}{\mathrm{d}s}\big(T\cos\theta - N\sin\theta\big) = 0$$

$$\frac{\mathrm{d}M}{\mathrm{d}s} = N$$

$$\frac{\mathrm{d}}{\mathrm{d}s}\big(T\sin\theta + N\cos\theta\big) = 0$$

Nonlinear beam theory

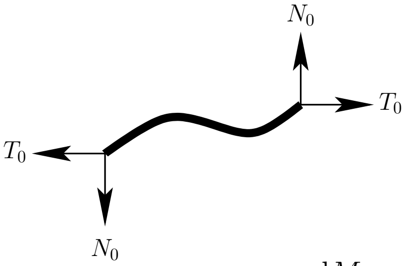
- Integrate with respect to s...

$$\boxed{T\cos\theta - N\sin\theta = T_0}$$

$=$ horizontal applied force

$$\boxed{T\sin\theta + N\cos\theta = N_0}$$

$=$ vertical applied force



- Invert to get...

$$\boxed{T = T_0\cos\theta + N_0\sin\theta} \qquad \boxed{N = N_0\cos\theta - T_0\sin\theta} = \frac{\mathrm{d}M}{\mathrm{d}s}$$

- Close with constitutive relation $\boxed{M = -B\dfrac{\mathrm{d}\theta}{\mathrm{d}s}}$ $\quad (B = EI)$
- Euler–Bernoulli beam equation

$$\boxed{B\frac{\mathrm{d}^2\theta}{\mathrm{d}s^2} + N_0\cos\theta - T_0\sin\theta = 0}$$

Nonlinear beam theory

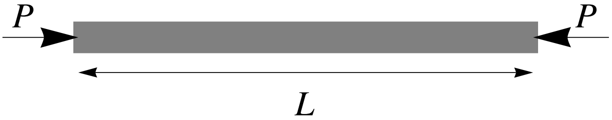
$$\boxed{B\frac{\mathrm{d}^2\theta}{\mathrm{d}s^2} + N_0\cos\theta - T_0\sin\theta = 0}$$

- Four BCs needed in general — two for 2^{nd} order ODE plus two more to determine T_0 and N_0.
- Examples:
  (a) Clamping — specify θ
  (b) Zero moment (simple support) — $\mathrm{d}\theta/\mathrm{d}s = 0$
  (c) Specify displacement — specify

$$X = \int_0^L \cos\theta\,\mathrm{d}s, \qquad Z = \int_0^L \sin\theta\,\mathrm{d}s$$

Weakly nonlinear theory and buckling

- Consider same setup as before: beam of length L subject to compressive force P and zero transverse force, clamped ends.



- Set $T_0 = -P$ and $N_0 = 0$ — nonlinear beam equation and clamped boundary conditions become

$$\boxed{\frac{\mathrm{d}^2\theta}{\mathrm{d}s^2} + \frac{P}{B}\sin\theta = 0} \qquad \boxed{\theta(0) = \theta(L) = 0}$$

- NB trivial solution $\theta(s) = 0$ always works.
- If $|\theta| \ll 1$ then $\sin\theta \sim \theta \longrightarrow$ same eigenvalue problem

$$\boxed{\theta(s) = A\sin\left(\frac{n\pi s}{L}\right)} \iff \boxed{\frac{PL^2}{\pi^2 B} = n^2} \quad (n = 1,\, 2,\, \cdots)$$

Weakly nonlinear theory and buckling

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}s^2} + \frac{P}{B}\sin\theta = 0 \qquad \theta(0) = \theta(L) = 0$$

- Now assume θ is small but not infinitesimal.
- Set $\theta = \delta\Theta$ with $0 < \delta \ll 1$.
- Also suppose applied compression is close to critical value, i.e.

  $\lambda = \dfrac{PL^2}{\pi^2 B} = 1 + \epsilon\lambda_1$ also with $0 < \epsilon \ll 1$.
- Also non-dimensionalise $s = L\xi$ so problem becomes...

$$\frac{\mathrm{d}^2\Theta}{\mathrm{d}\xi^2} + \pi^2(1+\epsilon\lambda_1)\frac{\sin(\delta\Theta)}{\delta} = 0 \qquad \Theta(0) = \Theta(1) = 0$$

- NB $\dfrac{\sin(\delta\Theta)}{\delta} \sim \Theta - \dfrac{\delta^2\Theta^3}{6} + \cdots$ as $\delta \to 0$

Weakly nonlinear theory and buckling

$$\frac{\mathrm{d}^2\Theta}{\mathrm{d}\xi^2} + \pi^2(1+\epsilon\lambda_1)\left(\Theta - \frac{\delta^2\Theta^3}{6} + \cdots\right) = 0$$

$$\Theta(0) = \Theta(1) = 0$$

- The problem contains two small parameters:
  - δ measures amplitude of beam deformation
  - ϵ measures excess loading
- To get a weakly nonlinear theory we balance these effects by choosing $\delta = \sqrt{\epsilon}$

$$\frac{\mathrm{d}^2\Theta}{\mathrm{d}\xi^2} + \pi^2\left[\Theta + \epsilon\left(\lambda_1\Theta - \frac{\Theta^3}{6}\right) + O\left(\epsilon^2\right)\right] = 0$$

- Now write solution as an asymptotic expansion $\Theta \sim \Theta_0 + \epsilon\Theta_1 + \cdots$ as $\epsilon \to 0$.

Weakly nonlinear theory and buckling

- Now equate coefficients of different powers of ϵ.
- At $O(1)$:

$$\boxed{\frac{\mathrm{d}^2\Theta_0}{\mathrm{d}\xi^2} + \pi^2\Theta_0 = 0} \qquad \boxed{\Theta_0(0) = \Theta_0(1) = 0}$$

- This is the problem we already solved: $\boxed{\Theta_0(\xi) = A_0 \sin(\pi\xi)}$ where A_0 is arbitrary
- At $O(\epsilon)$:

$$\boxed{\frac{\mathrm{d}^2\Theta_1}{\mathrm{d}\xi^2} + \pi^2\Theta_1 = \pi^2\left(\frac{\Theta_0^3}{6} - \lambda_1\Theta_0\right)} \qquad \boxed{\Theta_1(0) = \Theta_1(1) = 0}$$

- Homogeneous problem has nontrivial solution $\Theta_1(\xi) = \sin(\pi\xi)$
- By Fredholm Alternative inhomogeneous problem has no solution unless RHS satisfies solvability condition

Weakly nonlinear theory and buckling

- Solvability condition for Θ_1:

$$\boxed{\int_0^1 \left(\frac{\Theta_0^3}{6} - \lambda_1 \Theta_0\right) \sin(\pi\xi)\,\mathrm{d}\xi = 0}$$

- Plug in $\Theta_0(\xi) = A_0 \sin(\pi\xi)$ to get amplitude equation

$$\boxed{A_0\left(A_0^2 - 8\lambda_1\right) = 0}$$

- NB $\sin^3\Theta \equiv \dfrac{3}{4}\sin\Theta - \dfrac{1}{4}\sin(3\Theta)$
- If solvability condition is not satisfied, then it is impossible to satisfy the BCs $\Theta_1(0) = \Theta_1(1) = 0$.

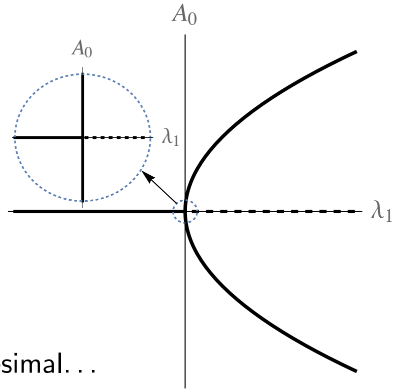
Weakly nonlinear theory and buckling

$$A_0\left(A_0^2 - 8\lambda_1\right) = 0$$

Response diagram

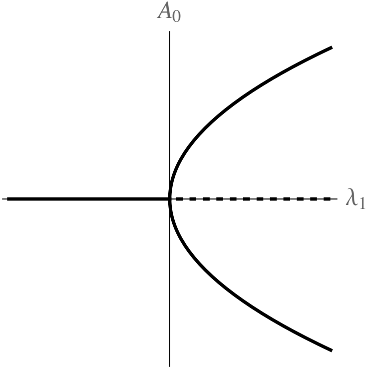
Pitchfork bifurcation

$\lambda_1 < 0 \;\Rightarrow\;$ one solution for A_0
$\lambda_1 > 0 \;\Rightarrow\;$ three solutions for A_0



- When amplitude A_0 is infinitesimal...
- Weakly nonlinear theory explains counterintuitive behaviour found before.

Weakly nonlinear theory and buckling



- NB pitchfork bifurcation can occur only when there is perfect symmetry.
- If there is small asymmetry (e.g. gravity) then...
- Always buckles downwards unless forced onto upper branch (Problem sheet 3).