Problem Sheet 0: Solutions

1. The derivation of the wave equation for the string is given in the Prelims course on Fourier Series & PDEs. In brief, we find the net vertical force on an element of length δx to be $T[\partial w/\partial x]_x^{x+\delta x}$. Equating this to "$ma$" for this element, $\rho \times \delta x \times \partial^2 w/\partial t^2$, we find that
$$\rho \frac{\partial^2 w}{\partial t^2} = T \frac{\partial^2 w}{\partial x^2},$$
which is the wave equation with wave speed $c = (T/\rho)^{1/2}$.

   Multiplying by $\partial w/\partial t$ and integrating between fixed points a and b, we have
$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\tfrac{1}{2}\int_a^b \left(\frac{\partial w}{\partial t}\right)^2 \mathrm{d}x\right] = \int_a^b T\frac{\partial^2 w}{\partial x^2}\frac{\partial w}{\partial t}\,\mathrm{d}x = \left[T\frac{\partial w}{\partial x}\frac{\partial w}{\partial t}\right]_a^b - \int_a^b T\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x \partial t}\,\mathrm{d}x.$$

   Rearranging we immediately have
$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\tfrac{1}{2}\int_a^b \left(\frac{\partial w}{\partial t}\right)^2 \mathrm{d}x + \tfrac{1}{2}\int_a^b T\left(\frac{\partial w}{\partial x}\right)^2 \mathrm{d}x\right] = \left[T\frac{\partial w}{\partial x}\frac{\partial w}{\partial t}\right]_a^b,$$
which is the desired result.

   Physically, this result represents the conservation of energy: the LHS gives the rate of change of the kinetic energy (first integral) and elastic stretching energy (second term). The RHS gives the rate of working of the vertical component of the tension force. Note, in particular, that when $w(a,t) = w(b,t) = 0$ (for example with clamping) then we recover the usual conservation of energy.

2. From the definitions, we have that

$$\delta\mathbf{x} = \delta\mathbf{X} + \mathbf{u}(\mathbf{X} + \delta\mathbf{X}) - \mathbf{u}(\mathbf{X})$$

or, in component form,

$$\delta x_i = \delta X_i + u_i(X_j + \delta X_j) - u_i(X_j) = \delta X_j \left(\delta_{ij} + \frac{\partial u_i}{\partial X_j} \right) + O(\delta X_j^2)$$

so that

$$\ell^2 = |\delta\mathbf{x}|^2 = \delta x_k \delta x_k = \delta X_j \left(\delta_{kj} + \frac{\partial u_k}{\partial X_j} \right) \left(\delta_{ki} + \frac{\partial u_k}{\partial X_i} \right) \delta X_i + O(|\delta\mathbf{X}|)^3$$

(Note that some care is needed to avoid conflicts in use of summation convention.)

Then

$$\ell^2 - L^2 \approx \delta X_i \delta X_j \left[\frac{\partial u_j}{\partial X_i} + \frac{\partial u_i}{\partial X_j} + \frac{\partial u_k}{\partial X_i}\frac{\partial u_k}{\partial X_j} \right]$$

since $L^2 = |\delta\mathbf{X}|^2 = \delta X_k \delta X_k$.

For small $|\partial u_i / \partial X_j|$, we can neglect the nonlinear term so that

$$\ell^2 - L^2 \approx 2 e_{ij} \delta X_i \delta X_j$$

where

$$e_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} \right)$$

Suppose we define new coordinates $\mathbf{x}' = \mathbf{P}\mathbf{x}$, where $\mathbf{P}$ is an orthogonal matrix. The displacement components with respect to the new axes clearly also satisfy $\mathbf{u}' = \mathbf{P}\mathbf{u}$. In component form, we can write

$$X_i' = P_{ik} X_k, \quad u_j' = P_{jl} u_l.$$

By orthogonality, we have $P_{ik}P_{jk} = P_{ki}P_{kj} = \delta_{ij}$ so that $X_k = P_{ik} X_i'$. Using the chain rule, we therefore have

$$\frac{\partial u_j'}{\partial X_i'} = P_{ik} \frac{\partial}{\partial X_k}(P_{jl} u_l) = P_{ik} P_{jl} \frac{\partial u_l}{\partial X_k}.$$

In dyadic notation, we can therefore write

$$(\nabla\mathbf{u})' = \mathbf{P} \cdot (\nabla\mathbf{u}) \cdot \mathbf{P}^T$$

so that $\nabla\mathbf{u}$ is a second rank tensor. [Note that $(\nabla\mathbf{u})_{ij} = \partial u_j / \partial x_i$, which is the displacement gradient.]

To show that $\mathcal{E}$ is a tensor, note that

$$(\nabla \mathbf{u}^T)' = \left[\mathbf{P} \cdot (\nabla \mathbf{u}) \cdot \mathbf{P}^T\right]^T = \mathbf{P} \cdot (\nabla \mathbf{u})^T \cdot \mathbf{P}^T$$

and so $\nabla \mathbf{u}^T$ is also a second rank tensor. Since $\mathcal{E} = (\nabla \mathbf{u} + \nabla \mathbf{u}^T)/2$, we conclude that $\mathcal{E}$ is also a tensor.

If $\boldsymbol{u}(\boldsymbol{X}) = \boldsymbol{c} + \boldsymbol{\omega} \wedge \boldsymbol{X}$ then we can write $u_i = c_i + \Omega_{ij} X_j$ where $\boldsymbol{\Omega}$ is skew-symmetric. Hence

$$e_{ij} = \frac{1}{2}\left[\frac{\partial}{\partial X_j}(c_i + \Omega_{ik} X_k) + \frac{\partial}{\partial X_i}(c_j + \Omega_{jl} X_l)\right] = \frac{1}{2}\left(\Omega_{ij} + \Omega_{ji}\right) = 0$$

(using the skew-symmetry of $\boldsymbol{\Omega}$).

Conversely, if $e_{ij} \equiv 0$, then from the diagonal elements we have

$$\frac{\partial u_1}{\partial X_1} = \frac{\partial u_2}{\partial X_2} = \frac{\partial u_3}{\partial X_3} = 0$$

so that

$$u_1 = u_1(X_2, X_3), \quad u_2 = u_2(X_1, X_3), \quad u_3 = u_3(X_1, X_2)$$

Examining other components, we have

$$0 = 2e_{12} = \frac{\partial u_1}{\partial X_2} + \frac{\partial u_2}{\partial X_1} \implies \frac{\partial^2 u_1}{\partial X_2^2} = 0 \tag{1}$$

$$0 = 2e_{13} = \frac{\partial u_1}{\partial X_3} + \frac{\partial u_3}{\partial X_1} \implies \frac{\partial^2 u_1}{\partial X_3^2} = 0 \tag{2}$$

$$0 = 2e_{23} = \frac{\partial u_2}{\partial X_3} + \frac{\partial u_3}{\partial X_2} \implies \frac{\partial^2 u_2}{\partial X_1 \partial X_3} + \frac{\partial^2 u_3}{\partial X_1 \partial X_2} = 0 \tag{3}$$

The last of these can be used with the first two to show that

$$0 = \frac{\partial}{\partial X_3}\left(-\frac{\partial u_1}{\partial X_2}\right) + \frac{\partial}{\partial X_2}\left(-\frac{\partial u_1}{\partial X_3}\right) - 2\frac{\partial^2 u_1}{\partial X_2 \partial X_3}$$

and so u_1 must be a linear function of X_2 and X_3, i.e.

$$u_1 = c_1 + a_1 X_2 + b_1 X_3.$$

Similarly,

$$u_2 = c_2 + a_2 X_3 + b_2 X_1$$

and

$$u_3 = c_3 + a_3 X_1 + b_3 X_2.$$

Examining the off-diagonal elements of $\mathcal{E}$ we find that

$$0 = 2e_{12} = a_1 + b_2 \implies b_2 = -a_1 := \omega_3 \tag{4}$$
$$0 = 2e_{13} = b_1 + a_3 \implies b_1 = -a_3 := \omega_2 \tag{5}$$
$$0 = 2e_{23} = a_2 + b_3 \implies b_3 = -a_2 := \omega_1 \tag{6}$$

from which we can immediately write $\boldsymbol{u}(\boldsymbol{X}) = \boldsymbol{c} + \boldsymbol{\omega} \wedge \boldsymbol{X}$ with $\boldsymbol{c} = (c_1, c_2, c_3)^T$, $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)^T$.

We have therefore shown that the strain is zero *if and only if* $\boldsymbol{u}$ is a rigid-body displacement, consisting of a translation $\boldsymbol{c}$ and a rotation $\boldsymbol{\omega} \wedge \boldsymbol{X}$.

3. The "proof" of the symmetry of $\mathcal{T} = (\tau_{ij})$ is in the printed lecture notes and does not need to be repeated here.

   Similarly, there is a "proof" in the lecture notes that the traction $\boldsymbol{\sigma}$ on a surface element with normal $\boldsymbol{n}$ is given by

$$\boldsymbol{\sigma} = \mathcal{T}\boldsymbol{n}.$$

   When we rotate the axes using an orthogonal matrix $\mathbf{P}$ the vectors $\boldsymbol{\sigma}$ and $\boldsymbol{n}$ transform according to

$$\boldsymbol{\sigma}' = \mathbf{P}\boldsymbol{\sigma}, \quad \boldsymbol{n}' = \mathbf{P}\boldsymbol{n},$$

   so if we set $\boldsymbol{\sigma}' = \mathcal{T}'\boldsymbol{n}'$ we get

$$\mathbf{P}\boldsymbol{\sigma} = \mathcal{T}'\mathbf{P}\boldsymbol{n} \implies \mathbf{P}\mathcal{T}\boldsymbol{n} = \mathcal{T}'\mathbf{P}\boldsymbol{n}.$$

   Since this is true for all $\boldsymbol{n}$, we deduce that

$$\mathbf{P}\mathcal{T} = \mathcal{T}'\mathbf{P} \implies \mathcal{T}' = \mathbf{P}\mathcal{T}\mathbf{P}^T$$

   and so $\mathcal{T}$ is a tensor.