	Mathematical Institute **University of Oxford** **Part C4.9 OT & PDE - Problem Sheet 1**

1. Suppose $f(t,\cdot)$ is a probability density on $\mathbb{R}$ for all times with zero mean and finite variance (second moment) solution to the granular flow equation

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial v}\left[f\left(\frac{\partial W}{\partial v} * f\right)\right],$$

with $W(v) = \frac{|v|^{\gamma+2}}{\gamma+2}$, $\gamma \geq 0$. Show that the variance is monotonically decreasing and converging to 0 as $t \to \infty$;

$$\int_{\mathbb{R}} |v|^2 f(t,v)dv \downarrow 0, \quad \text{as } t \to \infty,$$

by finding a suitable differential inequality. Moreover, show that there exists $T^* \leq \infty$ such that $f(t,\cdot) \overset{*}{\rightharpoonup} \delta_0$ weakly in the sense of measures as $t \to T^*$. In other words, show that for any test function $\phi \in C_b^\infty(\mathbb{R})$, we have

$$\int_{\mathbb{R}} f(t,v)\phi(v)dv \to \phi(0), \quad \text{as } t \to T^*.$$

2. Consider, for $m > 0$ the nonlinear diffusion equation given by

$$\frac{\partial u}{\partial t} = \Delta u^m, \quad x \in \mathbb{R}^d, t > 0.$$

By substituting $u(t,x) = t^{-d/\alpha} F(xt^{-1/\alpha})$, deduce that the self-similar profile F satisfies

$$\operatorname{div}(xF + \alpha \nabla F^m) = 0, \quad \text{when } \alpha = d(m-1)+2.$$

Find a smooth positive stationary probability solution for $0 < m \leq 1$ in one dimension.

3. Consider the self-similar Barenblatt profiles for $m > 0$

$$\mathscr{B}(t,x) = t^{-d/\alpha} F(xt^{-1/\alpha}), \quad F(\xi) = (C - \kappa|\xi|^2)_+^{\frac{1}{m-1}},$$

where $\alpha = d(m-1)+2$, $\kappa = \frac{m-1}{2m\alpha}$, and $C > 0$ is a constant that fixes unit mass $\int_{\mathbb{R}} \mathscr{B}(t,x)dx = 1$. Show that $\mathscr{B}(t,\cdot) \overset{*}{\rightharpoonup} \delta_0$ as $t \downarrow 0$.

4. Consider the following equations for $m > 0$

$$\frac{\partial u}{\partial t} = \Delta u^m, \quad x \in \mathbb{R}^d, t > 0, \tag{1}$$

and

$$\frac{\partial \rho}{\partial t} = \operatorname{div}(x\rho + \nabla \rho^m), \quad x \in \mathbb{R}^d, t > 0. \tag{2}$$

Show the following equivalence; if u solves (1), then

$$\rho(t,x) = e^{dt} u\left(\frac{e^{\alpha t}-1}{\alpha}, e^t x\right)$$

is a solution of (2). Similarly if ρ solves (2), then

$$u(t,x) = (1+\alpha t)^{-d/\alpha} \rho\left(\frac{1}{\alpha}\log(1+\alpha t), (1+\alpha t)^{-1/\alpha} x\right)$$

is a solution of (1). In the particular case of $m = 1$, can you find the solution to (2) obtained through the change of variables corresponding to the heat kernel solution to (1)? Can you interpret the result?

5. Consider $\mu \in \mathscr{P}(\mathbb{R}^d)$ and $T : \mathbb{R}^d \to \mathbb{R}^d$ measurable. Recall that the push-forward of μ through T is the measure $\nu = T\#\mu$ sending all measurable $B \subset \mathbb{R}^d$ to $\nu(B) = \mu(T^{-1}(B))$. We know the change of variables formula

$$\int_{\mathbb{R}^d} (\zeta \circ T)(x) d\mu(x) = \int_{\mathbb{R}^d} \zeta(y) d\nu(y), \quad \forall \zeta \in C_b(\mathbb{R}^d). \tag{3}$$

Show that (3) is also true for $\zeta \in L^1(\mathbb{R}^d)$ (recalling Part A Integration). Moreover, if T is a diffeomorphism and both $\mu, \nu \ll \mathscr{L}$ with corresponding densities $f, g \in L^1(\mathbb{R}^d)$, show that

$$f(x) = \det(\nabla T(x)) g(T(x))$$

for all $x \in \mathbb{R}^d$.

6. Suppose $\{f_\alpha\}_{\alpha\in\Lambda}$ for some index set Λ is a family of l.s.c. functions $f_\alpha : \mathbb{R}^d \to \mathbb{R}$. Show that $f = \sup_{\alpha\in\Lambda} f_\alpha$ is also l.s.c. Suppose now that f_α has a common modulus of continuity, that is

$$|f_\alpha(x) - f_\alpha(y)| \le \omega(|x-y|) \text{ for all } x, y \in \mathbb{R}^d$$

and for all $\alpha \in \Lambda$. Then both the supremum and the infimum of these functions, $f = \inf_{\alpha\in\Lambda} f_\alpha$ and $f = \sup_{\alpha\in\Lambda} f_\alpha$, if well defined, satisfy the same inequality.

7. Suppose $f : \mathbb{R}^d \to [0, \infty)$ is l.s.c. and bounded from below. Show that it can be approximated by an increasing sequence of continuous and bounded functions f_n in $\mathbb{R}^d$.

8. Show one of the implication of Prokhorov's Theorem: tightness implies compactness, that is, every tight sequence μ_n in $\mathscr{P}(\mathbb{R}^d)$ has a weakly or narrowly convergent subsequence to a limiting probability measure.