UNIVERSITY OF OXFORD	**Mathematical Institute** **University of Oxford** **Part C4.9 OT & PDE - Problem Sheet 2**

1. Show that

$$S := \inf_{\Pi \in \mathscr{M}_+(\mathbb{R}^d \times \mathbb{R}^d)} \left(\int_{\mathbb{R}^d \times \mathbb{R}^d} (c(x,y) - \varphi(x) - \psi(y))\, d\Pi(x,y) \right),$$

is given by

$$S = \begin{cases} 0 & \text{if } \varphi(x) + \psi(y) \le c(x,y) \text{ on } \mathbb{R}^d \times \mathbb{R}^d \\ -\infty & \text{otherwise} \end{cases}.$$

As a consequence, show that the infimum problem within the supremum

$$\sup_{\varphi,\psi} \left\{ \int_{\mathbb{R}^d} \varphi\, d\mu + \int_{\mathbb{R}^d} \psi\, d\nu + \inf_{\Pi} \left(\int_{\mathbb{R}^d \times \mathbb{R}^d} (c(x,y) - \varphi(x) - \psi(y))\, d\Pi(x,y) \right) \right\},$$

can be expressed as a constraint on the pair of functions (φ, ψ), so that the $\sup - \inf$ problem can be rewritten as

$$\sup_{\varphi,\psi \in C_b(\mathbb{R}^d)} \left\{ \int_{\mathbb{R}^d} \varphi d\mu + \int_{\mathbb{R}^d} \psi d\nu \,:\, \varphi(x) + \psi(y) \le c(x,y) \right\}.$$

2. Recall I_* is the value of the infimum of the Kantorovich formulation of optimal transport

$$I_* = \min_{\Pi \in \Gamma(\mu,\nu)} \left\{ \int_{\mathbb{R}^d \times \mathbb{R}^d} c(x,y) d\Pi(x,y) \right\}.$$

Show that the relaxed dual formulation of the Kantorovich problem satisfies $J_* \le I_*$ where

$$J_* := \sup_{(\varphi,\psi) \in \Phi_c} J[\varphi,\psi], \quad \text{with} \quad J[\varphi,\psi] := \int_{\mathbb{R}^d} \varphi\, d\mu + \int_{\mathbb{R}^d} \psi\, d\nu$$

and

$$\Phi_c := \left\{ (\varphi,\psi) \in L^1(d\mu) \times L^1(d\nu) : \varphi(x) + \psi(y) \le c(x,y) \, \text{a.e. w.r.t. } \mu \times \nu \right\}.$$

3. In lectures, Theorem 2.5 proved that given two probability measures $\mu, \nu \in \mathscr{P}_2(\mathbb{R}^d)$, there exists maximizers $(\varphi_0, \psi_0) \in \Phi_c$ such that

$$J[\varphi_0, \psi_0] = J_* = \max_{(\varphi,\psi) \in \Phi_c} J[\varphi,\psi].$$

Finish the proof which asserts that the maximisers can be chosen as c-transforms; $(\varphi_0, \psi_0) = (\eta_0^{cc}, \eta_0^c)$ where $\eta_0 \in L^1(d\mu)$. Prove the additional statement that the Kantorovich potentials can be chosen of the form (φ_0, φ_0^c) with $\varphi_0 \in C(\mathbb{R}^d)$ and c-concave.

4. Let $\mu, \nu, \omega \in \mathscr{P}(\mathbb{R}^d)$ with $\Pi_1 \in \Gamma(\mu,\nu), \Pi_2 \in \Gamma(\nu,\omega)$ optimal transference plans given by Theorem 2.3. Lemma 2.3 implies the existence of a measure $\gamma \in \mathscr{P}(\mathbb{R}^{3d})$ such that $P_{12}\#\Pi_1, P_{23}\#\gamma = \Pi_2$. Show that $\Pi_3 := P_{13}\#\gamma$ with $P_{13}(x,y,z) = (x,z)$ for $x,y,z \in \mathbb{R}^d$ belongs to $\Gamma(\mu,\omega)$.

5. Given $f_1, f_2, g_1, g_2 \in \mathscr{P}_p(\mathbb{R}^d)$, $1 \le p < \infty$, and $\alpha \in [0,1]$, show

$$d_p^p(\alpha f_1 + (1-\alpha) f_2, \alpha g_1 + (1-\alpha) g_2) \le \alpha d_p^p(f_1, g_1) + (1-\alpha) d_p^p(f_2, g_2).$$

6. Show that d_p is weakly lower semicontinuous in each argument for $1 \le p < \infty$.

7. Given two probability measures $\mu, \nu \in \mathscr{P}(\mathbb{R})$, let F, G be their distribution functions, respectively. Let $\mathbb{X}, \mathbb{Y}$ be the pseudo-inverses of F, G, respectively. Verify the equality

$$\int_0^1 |\mathbb{X}(\eta) - \mathbb{Y}(\eta)| d\eta = \int_{\mathbb{R}} |F(x) - G(x)| dx.$$

8. Consider the one dimensional linear Fokker-Planck equation

$$\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial x}\left(\rho \frac{\partial V}{\partial x}\right) + \sigma \frac{\partial^2 \rho}{\partial x^2},$$

with V uniformly convex, $V'(x) \ge \lambda > 0$, and global minimum at zero. Compute formally the equation satified by the pseudoinverse of solutions to the Fokker-Planck equation and draw conclusions about the asymptotic behavior.