UNIVERSITY OF OXFORD	**Mathematical Institute** **University of Oxford** **Part C4.9 OT & PDE - Problem Sheet 3**

1. Suppose $\Phi : (t,x) \mapsto \Phi_t(x)$ is a map such that it is C^1 in $t \in (0,T)$ and C^0 in $t \in [0,T]$. Moreover, suppose for each t, the map $x \mapsto \Phi_t(x)$ is a diffeomorphism from $\mathbb{R}^d$ to itself with linear growth at infinity for $\Phi_t(x)$ and its time derivative: there exists $C(T) > 0$ such that

$$\max\left(|\Phi_t(x)|, \left|\frac{d}{dt}\Phi_t(x)\right|\right) \le C(T)(1+|x|) \qquad \text{for all } x \in \mathbb{R}^d\,, 0 \le t \le T\,.$$

   Show that when $\mu \in \mathscr{P}_p(\mathbb{R}^d)$, the push-forward measure $\rho(t) := \Phi_t\#\mu \in C([0,T], \mathscr{P}_p(\mathbb{R}^d))$. More precisely, show that

$$[0,T] \ni t \mapsto \Phi_t\#\mu \in \mathscr{P}_p(\mathbb{R}^d)$$

   is a continuous mapping.

2. Complete the Grönwall inequality argument of Lemma 3.2; given $\rho_i \in C([0,T], \mathscr{P}_1(\mathbb{R}^d)$ with associated velocity fields $u^i = u(\rho_i)$ and flow maps $\Phi_t^i = \Phi_t(\rho_i)$ for $i = 1,2$, we showed

$$|\Phi_t^1(x) - \Phi_t^2(x)| \le L\int_0^t |\Phi_s^1(x) - \Phi_s^2(x)|dx + L\int_0^1 d_1(\rho_1(s), \rho_2(s))ds.$$

   From this, deduce

$$|\Phi_t^1(x) - \Phi_t^2(x)| \le L\int_0^1 e^{L(t-s)} d_1(\rho_1(s), \rho_2(s))ds.$$

3. Consider two solutions $x_1(t), x_2(t)$ to the following ODE

$$\frac{d}{dt}x(t) = -\nabla V(x(t)), \quad t > 0$$

   where $V : \mathbb{R}^d \to \mathbb{R}$ is a C^2 function with bounded second derivatives such that $D^2V \ge \lambda I_d$ for some $\lambda > 0$. Show the following inequality

$$d_2(\delta_{x_1(t)}, \delta_{x_2(t)}) \le e^{-\lambda t} d_2(\delta_{x_1(0)}, \delta_{x_2(0)}).$$

4. Recall the setting of Theorem 3.2; Consider $\mu \in \mathscr{P}_1(\mathbb{R}^d)$, and $W \in C^2(\mathbb{R}^d)$ with bounded second derivative. Define $F : X \to X$ where X is the complete metric space $C([0,T], \mathscr{P}_1(\mathbb{R}^d)$ with the metric $\mathscr{D}_{1,T}$ and F maps $\rho \mapsto \Phi_t(\rho)\#\mu$ where $\Phi_t(\rho)$ is the associated flow map to the velocity field $u(\rho) = -\nabla W * \rho$. Show that $\tilde{\rho} = F(\rho)$ is the unique weak solution in X to the linear problem

$$\frac{\partial \tilde{\rho}}{\partial t} + \nabla \cdot [\tilde{\rho} u(\rho)] = 0.$$

5. Fill in the remaining details of Theorem 3.2. That is, taking the strict contraction property of the map F in Problem 4 for granted, extend the unique local solution for all times.

6. Assume that $W \in C^2(\mathbb{R}^d)$ is a symmetric function with bounded second derivative. Show that the empirical measure defined by $\mu^N = \frac{1}{N}\sum_{i=1}^N \delta_{X_t^i}$ where $N \in \mathbb{N}$ is fixed and for $i = 1,\dots,N$ each X_t^i solves

$$\frac{dX_t^i}{dt} = -\frac{1}{N}\sum_{i\neq j}^{N} \nabla W(X_t^i - X_t^j),$$

with initial data X_0^i, is the unique weak solution in $C([0,\infty),\mathscr{P}_1(\mathbb{R}^d))$ to

$$\frac{\partial \rho}{\partial t} = \nabla\cdot[\rho(\nabla W * \rho)]\,.$$

with initial data $\mu^N(0)$.

7. Using the contraction estimate of Theorem 3.4 (and also similar to Theorem 3.5) and assuming the restitution coefficient $e \in [0,1)$, show existence and uniqueness of solutions in $C([0,\infty),\mathscr{P}_2(\mathbb{R}^3))$ to

$$\frac{\partial f}{\partial t} = Q_e^+(f,f) - f,$$

subject to initial data $f_0 \in \mathscr{P}_2(\mathbb{R}^3)$. Hint: As in Theorem 3.2, think of this equation as a fixed point problem.

8. Consider the Boltzmann equation in one dimension introduced in Chapter 1, Section 1.1, given by

$$\frac{\partial f}{\partial t} = Q_{\bar r}(f,f),$$

where $0 \le \bar r \le 1/2$, the dissipative collision between particles reads

$$v' = (1-\bar r)v + \bar r w; \qquad w' = \bar r v + (1-\bar r)w\,,$$

and the collision operator is defined by duality as

$$<\varphi, Q_{\bar r}(f,f)> = \int_{\mathbb{R}}\int_{\mathbb{R}} f(v)f(w)\Big[\varphi(v') - \varphi(v)\Big]\,dv\,dw\,.$$

Show a contraction type estimate in d_2 for the gain part of the collision operator for probability measures with equal mean. Show stability of solutions in $C([0,\infty),\mathscr{P}_2(\mathbb{R}^d))$ for initial data in $\mathscr{P}_2(\mathbb{R}^d)$ assuming the existence and uniqueness of weak solutions. Hint: Follow similar strategy as in Section 3.3.