UNIVERSITY OF OXFORD	**Mathematical Institute** **University of Oxford** **Part C4.9 OT & PDE - Problem Sheet 4**

1. Fill in the rest of the details in the proof of Theorem 4.1. Suppose $\mu \ll \mathscr{L}$ and ν are probability measures on $\mathbb{R}^d$ with $T : \mathbb{R}^d \to \mathbb{R}^d$ an optimal transport map leading to the optimal cost for $d_2^2(\mu, \nu)$ so that $\nu = T\#\mu$. For $0 \le s < 1$ and $x \in \mathbb{R}^d$, define $T_s(x) = (1-s)x + sT(x)$. Show that the velocity field $u(s,x)$ defined by

$$\frac{dT_s(x)}{ds} = u(s, T_s(x))$$

is well-defined by proving $T_s(x)$ is invertible and Lipschitz for $0 \le s < 1$ since $T = \nabla\varphi$ for some convex φ.

2. Finish the details of Lemma 4.2 for the strictly convex claims. That is, given $V : \mathbb{R}^d \to \mathbb{R}$ and $W : \mathbb{R}^d \to \mathbb{R}$ both strictly convex, show that

$$\mathscr{V}[\mu] = \int_{\mathbb{R}^d} V(x) d\mu(x) \quad \text{and } \mathscr{W}[\mu] = \frac{1}{2}\int_{\mathbb{R}^d\times\mathbb{R}^d} W(x-y) d\mu(x) d\mu(y)$$

are strictly d_2-convex (for $\mathscr{W}$ you need to show this is true unless the geodesic joining the measures is a translation of a given measure).

3. Prove Lemma 4.3 which states; Given Λ a nonnegative symmetric matrix, define

$$v(t) = \det((1-t)I_d + t\Lambda)^{\frac{1}{d}}, \quad t \in [0,1].$$

Show that v is concave in $[0,1]$ and strictly concave unless $\Lambda = \lambda I_d$ for some $\lambda \ge 0$.

4. This question follows up the previous exercise and proves Theorem 4.2. Suppose $U : [0,\infty) \to \mathbb{R}$ is a $C([0,\infty),\mathbb{R}) \cap C^2((0,\infty),\mathbb{R})$ function with $U(0) = 0$ such that $(0,\infty) \ni s \mapsto s^d U(s^{-d})$ is convex and non-increasing. Show that

$$t \mapsto v(t)^d U(v(t)^{-d}), \quad t \in [0,1]$$

is a convex function. Moreover, for $\mu, \nu \ll \mathscr{L}$ (so that they can be connected by a geodesic $\mu_t = T_t\#\mu$ with $T_t(x) = (1-t)x + t\nabla\varphi(x)$), recall that the internal energy of $\mu_t = \rho_t\mathscr{L}$ can be written as

$$\mathscr{U}[\mu_t] = \int_{\mathbb{R}^d} U(\rho_t(x))dx = \int_{\mathbb{R}^d} U\left(\frac{\rho_0(x)}{D(x,t)^d}\right) D(x,t)^d dx,$$

where $D(x,t) = \det((1-t)I_d + tD^2\varphi(x))^{\frac{1}{d}}$. By considering the map

$$t \mapsto D(x,t)^d U(\rho_0(x) D(x,t)^{-d})$$

and using the first part of this question, show that the internal energy $\mathscr{U}$ is d_2-convex.

5. Repeat the procedure in the notes of using the dynamic interpretation of d_2 to formally compute the convexity properties of interaction energies $\mathscr{W}$ of probability measures. More precisely, suppose $\mu_s = \rho_s \mathscr{L}$ is a geodesic curve with the following optimality conditions

$$\begin{cases} \partial_s \rho + \nabla \cdot (\rho \nabla \psi) = 0 \\ \partial_s \psi + \frac{1}{2} |\nabla \psi|^2 = 0 \end{cases}, \quad \psi \in C_0^\infty([0,1] \times \mathbb{R}^d).$$

Use these equations and compute formally

$$\frac{d^2}{ds^2} \mathscr{W}(\rho_s) = \frac{d^2}{ds^2} \left\{ \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d} W(x-y) \rho_s(x) \rho_s(y) dx dy \right\}.$$

Following this computation, what conditions on W guarantee the d_2-convexity of $\mathscr{W}$?

6. Remind yourself of the general family of PDEs in the first chapter of the course. Show that they can be formally written as gradient flows in the following way

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho u) = 0 \quad \text{in} \quad (0,\infty) \times \mathbb{R}^d \\ u = -\nabla \frac{\delta \mathscr{F}}{\delta \rho} \end{cases},$$

where the free energy functional, $\mathscr{F}[\rho]$ is given as a sum of internal, potential, and interaction energies $\mathscr{F} = \mathscr{U} + \mathscr{V} + \mathscr{W}$. In particular, for the internal energy, find the suitable U related to P. Show that McCann's condition (Theorem 4.2) for the function U is satisfied if and only if P satisfies the following conditions defined through U for every $s > 0$

$$P(s) \geq 0, \quad \left(1 - \frac{1}{d}\right) P(s) \leq s P'(s), \quad s U''(s) = P'(s), \quad P(0) = 0.$$

7. Show the variational characterization of the implicit Euler scheme for a convex and lower semicontinuous energy function $E : \mathbb{R}^d \to R \cup \{\infty\}$

$$-\frac{x_{k+1} - x_k}{\Delta t} \in \partial E(x_{k+1}) \iff x_{k+1} = \operatorname{argmin}_{x \in \mathbb{R}^d} \left\{ \frac{1}{2\Delta t} |x - x_k|^2 + E(x) \right\}.$$

8. Given the energy functional,

$$\mathscr{E}[u] := \begin{cases} \frac{1}{2} \int_{\mathbb{R}^d} |\nabla u|^2 dx & u \in H^1(\mathbb{R}^d) \\ +\infty & \text{otherwise} \end{cases},$$

show that $\partial E(u) \neq \emptyset$ if and only if $\Delta u \in L^2(\mathbb{R}^d)$.