Problem Sheet 0 (Revision and Contraction Mapping Theorem)

This problem sheet should not be submitted and solutions will be handed out in the lectures in W2

QUESTION 1. **Revisions on Banach Spaces.** Which of the following spaces are Banach spaces? Please justify your answer.

(1) $C_c(\mathbb{R}) = \{u \in C(\mathbb{R})\colon \operatorname{supp}(u) \subset\subset \mathbb{R}\}$ equipped with the supremum norm $\|u\|_{sup} := \sup_{x\in\mathbb{R}} |u(x)|$.

(2) $C_V(\mathbb{R}) = \{u \in C(\mathbb{R})\colon u(x) \to 0 \text{ for } |x| \to \infty\}$ with the supremum norm $\|u\|_{sup}$.

(3) $C_b(\mathbb{R}) := \{u \in C(\mathbb{R})\colon u \text{ bounded}\}$ equipped with $\|u\| := \sup_{x\in\mathbb{R}} \frac{2+\sin(x)}{3+\cos(x)}|u(x)|$

[You may use that $(C_b(\mathbb{R}), \|\cdot\|_{sup})$ is a Banach space]

QUESTION 2. **Revision on Gronwall Lemma.** Let $f\colon [t_0, t_0+c] \to [0,\infty)$ be a continuous function such that there exists two non-negative constants α and β such that

$$f(t) \le \alpha + \beta \int_{t_0}^{t} f(s)\,ds \qquad \text{for all } t \in [t_0, t_0+c].$$

Show that

$$f(t) \le \alpha \exp \beta(t - t_0)$$

for all $t_0 \le t \le t_0 + c$.

QUESTION 3. **Revision on compact operators** Let X, Y, Z be Banach spaces and let $T_1 : X \to Y$, $T_2 : Y \to Z$ be continuous operators (not necessarily linear!) Prove:

(1) If T_1 is compact then $T_2 \circ T_1$ is compact

(2) If T_2 is compact and if $T_1(B) \subset Y$ is bounded for every bounded subset $B \subset X$ then $T_2 \circ T_1$ is compact. Why is the second condition needed?

QUESTION 4. **Error Estimates for the Contraction Mapping Theorem.** Let (X, d) be a complete metric space and let $T\colon X \to X$ be a contractive map with constant $\kappa < 1$. Given $x_0 \in X$ consider the sequence $x_{n+1} = Tx_n$, and $x = \lim_{n\to\infty} x_n$. Show that

(1) $d(x_n, x_{n+m}) \le \frac{\kappa^n}{1-\kappa} d(x_1, x_0)$

(2) $d(x_n, x) \le \frac{\kappa^n}{1-\kappa} d(x_1, x_0)$

(3) $d(x_{n+1}, x) \le \frac{\kappa}{1-\kappa} d(x_{n+1}, x_n)$

(4) $d(x_{n+1}, x) \le \kappa d(x_n, x)$

QUESTION 5. **Counter-examples** In the examples below, find which assumptions of the Contraction Mapping Theorem are satisfied and which are not, and show that the map f does not have a fixed point.

(1) $f(x) = \sqrt{x^2+1}$ on $X = [0, 1]$.

(2) $f(x) = \sqrt{x^2+1}$ on $X = [0, \infty)$.

(3) $f(x) = \frac{1}{2}\sin x$ on $X = (0, \pi/4]$.