Problem Sheet 4

Let X be a separable reflexive Banach space and $M \subset X$ be non-empty, closed and convex.

QUESTION 1. **Monotone Operators** Let $A\colon M \to X^*$ be a monotone operator.

(1) Using monotonicity first, and then Minty's Lemma, show that A satisfies condition (H3).
(2) Show that if A is strictly monotone, i.e. so that $\langle A(u) - A(v), u - v\rangle > 0$ for all $u \neq v$, then there exists at most one solution $u \in M$ of the variational inequality $\langle A(u), u - v\rangle \leq 0$ for all $v \in M$.
(3) Show that if A is strongly monotone, i.e. so that there exists $c > 0$ such that

$$\langle A(u) - A(v), u - v\rangle > c \left\| u - v \right\|^2 ,$$

and A maps bounded sets to bounded sets, then the variational inequality has a unique solution.
(4) Using the above show that the equation

$$-\Delta u = f \text{ in } \Omega, \qquad u = 0 \text{ on } \partial\Omega$$

has a unique solution for every $f \in L^2(\Omega)$, $\Omega \subset \mathbb{R}^n$ a bounded open subset with smooth boundary.

QUESTION 2. **Monotonicity, Convexity**

Let X be a Banach space and $F\colon X \to \mathbb{R}$ Gâteaux differentiable in every point $u \in X$ with Gâteaux derivative $F'(u)$. Show that

$$F \text{ is convex} \quad \Leftrightarrow \quad F'\colon X \to X^* \text{ is monotone.}$$

Remark:

- A map $G\colon X \to X^*$ is monotone if $\langle G(u) - G(v), u - v\rangle \geq 0$ for all $u, v \in X$ (i.e. hemicontinuity, as in the definition of a monotone operator, is not required).
- A function $F\colon X \to \mathbb{R}$ is convex on X, if $F(tu + (1-t)v) \leq tF(u) + (1-t)F(v)$ for all $t \in [0,1]$ and $u, v \in X$.
- Recall that a differentiable function $g : I \subset \mathbb{R} \to \mathbb{R}$ is convex on I if g' is monotonically increasing on I. Consider $g(t) := F(tu + (1-t)v)$.

QUESTION 3. **Strongly monotone operator** Let $\Omega = (-1, 1)$ and $X = H^2(\Omega) \cap H^1_0(\Omega)$ endowed with the H^2-norm.

(a) Let $A\colon X \to X^*$ be defined via

$$\langle A(u), v\rangle := \int_\Omega u'' v'' dx.$$

Show that A is a strongly monotone operator, i.e. hemicontinuous and so that there exists some $c_0 > 0$ with

$$\langle A(u) - A(v), u - v\rangle \geq c_0 \|u - v\|^2 \quad \text{for all } u, v \in M.$$

Hint: Use Poincaré's inequality, as well as Poincaré's inequality for functions with mean value zero.

(b) Let now $F_\mu(u) := A(u) + \mu B(u)$ where $B(u)(v) := u(0) \cdot v(0) + \int_\Omega x \cdot v(x) dx$.

Show that $F_\mu : X \to X^*$ is well defined for any $\mu \in \mathbb{R}$ and that there exists a number $\mu_0 > 0$ so that for each μ with $|\mu| \leq \mu_0$ there exists a unique solution of the equation

$$F_\mu(u) = 0.$$

(c) Let now $\mu \geq 0$. Determine a functional $I_\mu : X \to \mathbb{R}$ so that the following holds: $u \in X$ is a solution of $F_\mu(u) = 0$ if and only if u is a minimiser of I_μ on X

QUESTION 4. Consider a domain $\Omega \subset \mathbb{R}^n$ which is smooth and bounded, and $g \in C^2(\mathbb{R}^n)$ such that $g \leq 0$ on $\partial\Omega$. Consider the energy I given by

$$I(v) = \int_\Omega |\Delta v|^2 + fv dx,$$

for some $f \in L^2(\Omega)$.

(1) Find the Euler-Lagrange equation satisfied by the critical points of $I(v)$ and prove that every critical point of I is a minimiser.

(2) Consider the set M given by

$$M := \left\{ v \in H^2(\Omega) \cap H_0^1(\Omega) \,|\, v \geq g \text{ a.e. on } \Omega \right\}.$$

Show that there exists a unique minimizer of I on M —check carefully that the assumptions of the Theorem(s) you use are satisfied. You may use without proof that for all $u \in H_0^1(\Omega) \cap H^2(\Omega)$

$$\|u\|_{H_0^1(\Omega)} \leq C \|\Delta u\|_{L^2(\Omega)},$$

where the constant C is independent of u.

QUESTION 5. **Three approaches to the same problem**. Consider a domain $\Omega = \{(x, y) \in \mathbb{R}^2 \text{ s.t.} x^2 + y^2 \leq 1\}$ and the equation

$$-\Delta u + u^5 = 1 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega.$$

- Show that this equation makes sense in $H_0^1(\Omega)$, that is, it has a legitimate weak variational formulation.
- Using the first part of the course, show that you can formulate it as a fixed point problem of the form $u = T(u)$ where T is a continuous compact map.
- Find a simple subsolution $\underline{u}$ and a simple supersolution $\bar{u}$. Show that the problem can be transformed into
$$-\Delta u + \lambda u = f_\lambda(u)$$
for a constant $\lambda > 0$ chosen so that $f_\lambda(u)$ is increasing when $\underline{u} \leq u \leq \bar{u}$, and use the method of sub and super solutions to show that a solution u can be found by a constructive (iterative) method.
- Using Schauder's FPT and the above show that there exists a solution.
- Use the variational inequality approach to find a solution in $H_0^1(\Omega)$.
- What can you say about uniqueness?