# C3.11 Riemannian Geometry

## Problem Sheet 1

## Hilary Term 2020–2021

This problem sheet is based on Lectures 1–4.

1. Let

$$\mathcal{H}^n = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : \sum_{j=1}^{n} x_j^2 - x_{n+1}^2 = -1,\ x_{n+1} > 0\}$$

and let $g$ be the restriction of

$$h = \sum_{j=1}^{n} \mathrm{d}x_j^2 - \mathrm{d}x_{n+1}^2$$

on $\mathbb{R}^{n+1}$ to $\mathcal{H}^n$.

(a) Show that $g$ is a Riemannian metric on $\mathcal{H}^n$.

(b) Let $f(x) = Ax$ be a linear map on $\mathbb{R}^{n+1}$ given by $A = (a_{ij}) \in M_{n+1}(\mathbb{R})$ and let

$$G = \begin{pmatrix} I_n & 0 \\ 0 & -1 \end{pmatrix}$$

where $I_n$ is the $n \times n$ identity matrix. Show that $f$ defines an isometry on $(\mathcal{H}^n, g)$ if and only if

$$A^{\mathrm{T}} G A = G \quad \text{and} \quad a_{n+1,n+1} > 0.$$

2. Let $(M, g)$ be a connected Riemannian manifold and let $\widetilde{M}$ be the universal cover of $M$.

(a) Show that there exists a unique Riemannian metric $\tilde{g}$ on $\widetilde{M}$ such that the covering map $\pi : (\widetilde{M}, \tilde{g}) \to (M, g)$ is a local isometry.

(b) Show that the fundamental group of $M$ acts on $(\widetilde{M}, \tilde{g})$ by isometries.

3. Let $(M_1, g_1)$ and $(M_2, g_2)$ be Riemannian manifolds with Levi-Civita connections $\nabla_1$ and $\nabla_2$ respectively. Recall that $T_{(p_1,p_2)}(M_1 \times M_2) \cong T_{p_1}M_1 \times T_{p_2}M_2$ for all $(p_1, p_2) \in M_1 \times M_2$. Define $g$ on $M_1 \times M_2$ by

$$g_{(p_1,p_2)}((X_1, X_2), (Y_1, Y_2)) = (g_1)_{p_1}(X_1, Y_1) + (g_2)_{p_2}(X_2, Y_2).$$

(a) Show that $g$ is a Riemannian metric on $M_1 \times M_2$.

(b) Show that the Levi-Civita connection $\nabla$ of $g$ on $M_1 \times M_2$ satisfies

$$\nabla_{(X_1,X_2)}(Y_1, Y_2) = \big((\nabla_1)_{X_1} Y_1, (\nabla_2)_{X_2} Y_2\big)$$

for all vector fields $(X_1, X_2)$, $(Y_1, Y_2)$ on $M_1 \times M_2$.

4. Let $(H^2, h)$ be the upper half-space with the hyperbolic metric

$$h = \frac{\mathrm{d}x_1^2 + \mathrm{d}x_2^2}{x_2^2}.$$

(a) Calculate the Christoffel symbols of $h$ in the coordinates $(x_1, x_2)$ on $H^2$ using the definition or formula for the Christoffel symbols.

Let $\alpha : [0, L] \to (H^2, h)$ be the curve $\alpha(t) = (t, 1)$ and let $\tau_\alpha$ be the parallel transport along $\alpha$.

(b) Let $X_0 = \partial_2 \in T_{(0,1)}H^2$. Calculate $\tau_\alpha(X_0)$ and show that, viewed as a vector in Euclidean $\mathbb{R}^2$, it makes an angle $L$ with the vertical axis.

Let

$$G = \{u : \mathbb{R} \to \mathbb{R} : u(x_1, x_2)(t) = x_1 + tx_2, x_1 \in \mathbb{R},\ x_2 > 0\}$$

and define a manifold structure on $G$ so that $f : G \to H^2$ given by $f(u(x_1, x_2)) = (x_1, x_2)$ is a diffeomorphism. Define a Riemannian metric $g$ on $G$ by $g = f^*h$.

(c) Show that, for all $v \in G$, the map $L_v : G \to G$ given by $L_v(u) = v \circ u$ is an isometry of $g$.

5. Let $\mathcal{S}^2$ be the unit sphere in $\mathbb{R}^3$ endowed with the round metric $g$, let $U = \mathcal{S}^2 \setminus \{(0, 0, 1)\}$ and let $\varphi : U \to \mathbb{R}^2$ be

$$\varphi(x_1, x_2, x_3) = \frac{(x_1, x_2)}{1 - x_3}$$

so that

$$\varphi^{-1}(y_1, y_2) = \frac{(2y_1, 2y_2, y_1^2 + y_2^2 - 1)}{y_1^2 + y_2^2 + 1}$$

(a) Show that

$$(\varphi^{-1})^* g = \frac{4(\mathrm{d}y_1^2 + \mathrm{d}y_2^2)}{(1 + y_1^2 + y_2^2)^2}.$$

Let $\beta : [0, 2\pi] \to \mathbb{R}^2$ be given by $\beta(t) = (\cos t, \sin t)$.

(b) Using the fact that $\varphi^{-1} : (\mathcal{S}^2 \setminus \{0, 0, 1\}, g) \to (\mathbb{R}^2, (\varphi^{-1})^* g)$ is an isometry or otherwise, show that the restrictions of the vector fields

$$y_1\partial_1 + y_2\partial_2 \quad \text{and} \quad -y_2\partial_1 + y_1\partial_2$$

to $\beta$ are parallel along $\beta$ with respect to the metric $(\varphi^{-1})^* g$.