C3.11 Riemannian Geometry

Problem Sheet 2

Hilary Term 2020–2021

This problem sheet is based on Lectures 4–8a.

1. Let X, Y be vector fields on (M, g). Let $p \in M$ and let $\alpha : (-\epsilon, \epsilon) \to M$ be the integral curve of X with $\alpha(0) = p$. For all $t \in (-\epsilon, \epsilon)$ let $\tau_t : T_pM \to T_{\alpha(t)}M$ be parallel transport along $\alpha|_{[0,t]}$. Show that
$$\nabla_X Y(p) = \frac{\mathrm{d}}{\mathrm{d}t}\Big(\tau_t^{-1}\big(Y(\alpha(t))\big)\Big)|_{t=0}.$$

2. The Euclidean Schwarzschild metric (of mass $m > 0$) is defined for $(\cos\frac{t}{4m}, \sin\frac{t}{4m}) \in \mathcal{S}^1$, $r > 2m$, $\theta \in (0, \pi)$ and $(\cos\phi, \sin\phi) \in \mathcal{S}^1$ by
$$g = \left(1 - \frac{2m}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2m}{r}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\phi^2$$
and extends smoothly to $\theta = 0, \pi$.

   (a) Show that there are no geodesics in this metric with r constant.

   (b) Show that, given any point p with $r > 2m$ there exists a finite length geodesic γ starting at p ending at a point q with $r = 2m$.

3. Let (M, g) be an n-dimensional Riemannian manifold. Let $p \in M$ and let U be a normal neighbourhood of p.

   Let $\{E_1, \ldots, E_n\}$ be an orthonormal basis for T_pM, let $\psi : T_pM \to \mathbb{R}^n$ be given by $\psi(\sum_{i=1}^n x_iE_i) = (x_1, \ldots, x_n)$ and let $\varphi = \psi \circ \exp_p^{-1} : U \to \mathbb{R}^n$.

   (a) Let $\gamma(t)$ be a geodesic through p in U in M. Show that
$$\varphi \circ \gamma(t) = (a_1t, \ldots, a_nt)$$
for $(a_1, \ldots, a_n) \in \mathbb{R}^n$.

   (b) Show that in (U, φ), we have $g_{ij}(p) = \delta_{ij}$ and $\Gamma_{ij}^k(p) = 0$.

   (c) Hence, or otherwise, show that there is open set $V \ni p$ and orthonormal vector fields $E_1, \ldots, E_n$ on V such that
$$\nabla_{E_i}E_j(p) = 0.$$

4. Let (M, g) be a Riemannian manifold. Recall that a *Killing field* on M is a vector field X such that $\mathcal{L}_X g = 0$ or, equivalently, that the flow of X near any point consists of local isometries.

   (a) Let $p \in M$ and let U be a normal neighbourhood of p. Suppose that X is a Killing field on (M, g) so that $X(p) = 0$ and $X(q) \neq 0$ for all $q \in U \setminus \{p\}$.

   By using the First variation formula, or otherwise, show that X is tangent to all sufficiently small geodesic spheres centred at p.

   (b) Show that X is a Killing field on (M, g) if and only if, for all vector fields Y, Z on M,

$$g(\nabla_Y X, Z) + g(\nabla_Z X, Y) = 0.$$

   [*Hint: To deduce the equation, you may want to restrict to neighbourhoods of points p where $X(p) \neq 0$ and consider coordinates so that X is a coordinate vector field which is normal at p to a hypersurface in M containing p.*]

5. Let (M, g) be a Riemannian manifold, let $f : M \to \mathbb{R}$ be a smooth function and let X be a vector field on M.

   (a) Note that we have a linear map from vector fields to vector fields given by $Y \mapsto \nabla_Y X$. We define the *divergence* of X to be the smooth function

$$\operatorname{div} X = \operatorname{tr}(Y \mapsto \nabla_Y X).$$

   Show that if X is a Killing field then $\operatorname{div} X = 0$.

   (b) Recall that $Y \mapsto g(Y, .)$ defines an isomorphism between vector fields and 1-forms on M. We define the *gradient* of f to be the vector field ∇f given by

$$g(\nabla f, .) = \mathrm{d}f.$$

   We define the *Laplacian* of f to be the smooth function

$$\Delta f = \operatorname{div} \nabla f.$$

   Show that

$$\Delta(f^2) = 2f\nabla f + 2|\nabla f|^2.$$

   Now suppose further that M is compact, connected and oriented with Riemannian volume form Ω.

   (c) Show that

$$\mathcal{L}_X \Omega = (\operatorname{div} X)\Omega.$$

   Relate this to the result about Killing fields from (a).

   (d) Show that if $\Delta f \geq 0$ on M then f is constant.