C3.11 Riemannian Geometry

Problem Sheet 3

Hilary Term 2020–2021

This problem sheet is based on Lectures 8b–12b.

1. Let E_1, E_2, E_3 be vector fields on $\mathcal{S}^3$ such that $[E_i, E_j] = -2\epsilon_{ijk}E_k$. For $\lambda > 0$, let

$$X_1 = \lambda E_1, \quad X_2 = E_2, \quad X_3 = E_3$$

and define a Riemannian metric g on $\mathcal{S}^3$ by the condition that

$$g(X_i, X_j) = \delta_{ij}$$

   (a) Show that $(\mathcal{S}^3, g)$ is Einstein if and only if $\lambda = 1$.

   (b) Find a necessary and sufficient condition on λ so that the scalar curvature of $(\mathcal{S}^3, g)$ is zero.

2. Let $(\mathcal{S}^n, g)$ be the round n-sphere and let h be the product metric on $\mathcal{S}^n \times \mathcal{S}^n$.

   Show that $(\mathcal{S}^n \times \mathcal{S}^n, h)$ is Einstein with non-negative sectional curvature.

3. Let M be $\mathrm{SO}(n)$, $\mathrm{O}(n)$, $\mathrm{SU}(m)$ or $\mathrm{U}(m)$ and let g be the bi-invariant metric on M given by

$$g_A(B, C) = -\operatorname{tr}(A^{-1}BA^{-1}C)$$

for all $A \in M$ and $B, C \in T_AM$. Let $L_A : M \to M$ denote left-multiplication by A and let

$$\mathcal{X} = \{\text{vector fields } X \text{ on } M \, : \, (L_A)_*X = X \, \forall A \in M\}.$$

   (a) Show that, for all $X, Y \in \mathcal{X}$,

$$\nabla_X Y = \frac{1}{2}[X, Y].$$

   [*You may assume that* $[X, Y](I)$ *is the matrix commutator of* $X(I)$ *and* $Y(I)$*, where* I *is the identity matrix.*]

   (b) Show that the sectional curvatures of (M, g) are non-negative and that (M, g) is flat if and only if $n = 2$ or $m = 1$.

4. (a) Show that an oriented minimal hypersurface in $(\mathbb{R}^{n+1}, g_0)$ is flat if and only if it is totally geodesic.

   (b) Let

$$M = \{(z_1, z_2) \in \mathbb{C}^2 \, : \, |z_1| = |z_2| = \frac{1}{\sqrt{2}}\} \subseteq \mathcal{S}^3$$

   and let g be the induced metric on M from the round metric on $\mathcal{S}^3$.

   Show that (M, g) is flat and that M is a minimal hypersurface in $\mathcal{S}^3$ which is not totally geodesic.

5. (a) Let $\gamma : [0, L] \to (M, g)$ be a geodesic and let $f : (-\epsilon, \epsilon) \times [0, L] \to M$ be a variation of γ so that the curve $\gamma_s : [0, L] \to (M, g)$ given by $\gamma_s(t) = f(s, t)$ is a geodesic for all $s \in (-\epsilon, \epsilon)$. Show that the variation field V_f of f is a Jacobi field along γ.

(b) Let

$$\mathcal{H}^n = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : \sum_{i=1}^{n} x_i^2 - x_{n+1}^2 = -1,\ x_{n+1} > 0\}$$

and let g be the restriction of $h = \sum_{i=1}^{n} \mathrm{d}x_i^2 - \mathrm{d}x_{n+1}^2$ on $\mathbb{R}^{n+1}$ to $\mathcal{H}^n$. Given that the normalized geodesics γ in $(\mathcal{H}^n, g)$ with $\gamma(0) = x$ and $\gamma'(0) = X$ are given by

$$\gamma(t) = x \cosh s + X \sinh s,$$

show that $(\mathcal{H}^n, g)$ has constant sectional curvature -1.

6. **(Optional.)** Let $(\mathcal{S}^{2n+1}, g)$ be the round $(2n+1)$-sphere, view $\mathcal{S}^{2n+1} \subseteq \mathbb{C}^{n+1}$ and let $\pi : \mathcal{S}^{2n+1} \to \mathbb{CP}^n$ be the projection map. For $z \in \mathcal{S}^{2n+1}$ we have $E(z) = iz$ (identifying tangent vectors in $\mathbb{C}^n$ with $\mathbb{C}^n$), $\ker \mathrm{d}\pi_z = \mathrm{Span}\{E(z)\}$ and we let

$$H_z = \{X \in T_z\mathcal{S}^{2n+1} : g(X, E(z)) = 0\} \quad \text{and} \quad \Phi_z = \mathrm{d}\pi_z : H_z \to T_{\pi(z)}\mathbb{CP}^n.$$

The Fubini–Study metric h on $\mathbb{CP}^n$ is then given by

$$h_{\pi(z)}(X, Y) = g_z(\Phi_z^{-1}(X), \Phi_z^{-1}(Y)).$$

(a) For any vector field X on $\mathbb{CP}^n$ we define a vector field $\widehat{X}$ on $\mathcal{S}^{2n+1}$ by

$$\widehat{X}(z) = \Phi_z^{-1}\big(X(\pi(z))\big).$$

If $\widehat{\nabla}$ is the Levi-Civita connection of g and ∇ is the Levi-Civita connection of h, show that, for all vector fields X, Y on $\mathbb{CP}^n$

$$\widehat{\nabla}_{\widehat{X}}\widehat{Y} = \widehat{\nabla_X Y} + \frac{1}{2} g([\widehat{X}, \widehat{Y}], E)E.$$

[Hint: Show that $[\widehat{X}, \widehat{Y}] - \widehat{[X, Y]}$ and $[\widehat{X}, E]$ are multiples of E.]

(b) Show that $\gamma : (-\epsilon, \epsilon) \to (\mathbb{CP}^n, h)$ is a geodesic with $\gamma(0) = \pi(z)$ if and only if $\gamma = \pi \circ \hat{\gamma}$ where $\hat{\gamma} : (-\epsilon, \epsilon) \to (\mathcal{S}^{2n+1}, g)$ is a geodesic with $\hat{\gamma}(0) = z$ and $\hat{\gamma}'(0) \in H_z$.

(c) Since $X \in H_z$ if and only if $iX \in H_z$, we can define $J = J_{\pi(z)} : T_{\pi(z)}\mathbb{CP}^n \to T_{\pi(z)}\mathbb{CP}^n$ by

$$J(X) = \mathrm{d}\pi_z(i\Phi_z^{-1}(X)),$$

which then extends to a map J from vector fields to vector fields on $\mathbb{CP}^n$. Let $X, Y \in T_{\pi(z)}\mathbb{CP}^n$ be orthogonal unit vectors and write $Y = \cos\alpha Z + \sin\alpha JX$ where Z is orthogonal to JX and unit length. Show that the sectional curvature K of $(\mathbb{CP}^n, h)$ satisfies

$$K(X, Y) = 1 + 3\sin^2\alpha.$$

[Hint: Let γ be a geodesic in $(\mathbb{CP}^n, h)$ with $\gamma(0) = \pi(z)$ and $\gamma'(0) = X$, and consider a variation $f(s, t)$ of γ so that $\gamma_s(t) = f(s, t)$ is geodesic for all s such that $\gamma_s(0) = \pi(z)$ and $\gamma_s'(0) = \cos sX + \sin sY$. You may want to consider the cases $\sin\alpha = 0$ and $\cos\alpha = 0$ first.]