C3.11 Riemannian Geometry

Problem Sheet 4

Hilary Term 2020–2021

This problem sheet is based on Lectures 12c–16.

1. Let $f : (M, g) \to (N, h)$ be a surjective local isometry.

   (a) Show that if (M, g) is complete then (N, h) is complete.

   (b) If (N, h) is complete, is (M, g) complete? Give a proof or a counterexample.

   Let $(\widetilde{M}, \tilde{g})$ be the universal cover of (M, g) with the covering metric.

   (c) Show that $(\widetilde{M}, \tilde{g})$ is complete if and only if (M, g) is complete.

2. Let B^n be the unit ball in $\mathbb{R}^n$ and let

$$g = \frac{4\sum_{i=1}^n \mathrm{d}x_i^2}{(1 - \sum_{i=1}^n x_i^2)^2}.$$

   By considering normalized geodesics in (B^n, g) through 0, show that (B^n, g) is complete.

3. Let (N, g) be an oriented $(n+1)$-dimensional Riemannian manifold. Let $f : N \to \mathbb{R}$ be a smooth function and let $h = e^{2f}g$.

   (a) Let ∇^g and ∇^h be the Levi-Civita connections of g and h. Show that

$$\nabla^h_X Y = \nabla^g_X Y + X(f)Y + Y(f)X - g(X, Y)\nabla^g f$$

   for all vector fields X, Y on N.

   (b) Let M be a connected oriented hypersurface in (N, g) with unit normal vector field ν so that the shape operator satisfies

$$S_\nu = \lambda\,\mathrm{id}$$

   for a smooth function $\lambda : M \to \mathbb{R}$.

   Show that the shape operator of M in (N, h) satisfies

$$S_{e^{-f}\nu} = \mu\,\mathrm{id}$$

   for a smooth function $\mu : M \to \mathbb{R}$ which should be identified in terms of λ and f.

   Now let $R > 0$, let

$$M = \{(x_1, \ldots, x_{n+1}) \in H^{n+1} : \sum_{i=1}^{n+1} x_i^2 = R^2\}$$

   with its standard orientation and let h be the hyperbolic metric on H^{n+1}.

   (c) Calculate the mean curvature and sectional curvatures of M in (H^{n+1}, h) with its induced metric.

4. (a) Let $(M.g)$ be a complete Riemannian manifold with non-positive sectional curvature, let p, q be points in M and let α be a curve in M from p to q.

   Show that there is a unique geodesic γ in (M, g) from p to q which is homotopic to α.

   (b) Let (M, g) be an oriented even-dimensional manifold with positive sectional curvature and let $\gamma : \mathcal{S}^1 \to (M, g)$ be a closed geodesic.

   Show that there is a closed curve $\alpha : \mathcal{S}^1 \to (M, g)$ homotopic to γ such that $L(\alpha) < L(\gamma)$.

5. (a) Let $n, m \in \mathbb{N}$. Show that $\mathcal{S}^n \times \mathcal{S}^m$ admits a Riemannian metric of positive Ricci curvature if and only if $n \geq 2$ and $m \geq 2$.

   (b) Let G be a connected Lie group with identity e which admits a bi-invariant Riemannian metric. Suppose that the centre of the Lie algebra $\mathfrak{g} = T_eG$ is trivial.

   Show that G and its universal cover are compact, and hence that $\mathrm{SL}(n, \mathbb{R})$ does not admit a bi-invariant metric for $n \geq 2$.

   [*You may assume that the results of Problem sheet 3 Question 3 extend to any Lie group with a bi-invariant Riemannian metric.*]

   (c) Show that $\mathbb{RP}^2 \times \mathbb{RP}^2$ does not admit a Riemannian metric of positive sectional curvature.

   [*Hint: You may want to think about the orientable double cover.*]