Elliptic Curves. HT 2020/21. Sheet 2.

1. Let K be a field with non-Archimedean valuation $|\ |$.
(a). For any $x, y \in K$ show that, if $|x| \neq |y|$ then $|x \pm y| = \max(|x|, |y|)$.
(b). If $x_1, \ldots, x_n \in K$ and if there exists ℓ such that $|x_\ell| > |x_i|$ for all $i \neq \ell$, then show that $|x_1 + \ldots + x_n| = |x_\ell|$.

(c). Suppose that $s_n \to s$ in $K, |\ |$. Show that $|s_n| \to |s|$ in $\mathbb{R}, |\ |_\infty$. When $s \neq 0$, show that there exists N such that, for all $n > N$, $|s_n| = |s|$.

2(a). Find: $|3/50|_5$, $|3/50|_3$, $|3/50|_7$, $d_5(2/3, 1/5)$, $d_7(2/3, 1/5)$, $d_{11}(2/3, 1/5)$.
(b). Describe $|3/7|_p$ for all p. What is the product $\prod |3/7|_i$, taken over $i = p$, for all primes p, and $i = \infty$? Given any $x \in \mathbb{Q}$ ($x \neq 0$), what is $\prod |x|_i$?

3. Which of the following are convergent in $\mathbb{Q}_5$?
(**a**). $1/5^n$. (**b**). n. (**c**). $n!$ (**d**). $3 + 10^n$. (**e**). $\sum_0^\infty 10^n$. (**f**). $\sum_0^\infty 7^n$.

4. For each p, m, r, either find an $x \in \mathbb{Z}$ such that $|x^2 - r|_p \leqslant p^{-m}$ or show that no such x exists.

(a). $p = 5, r = -1, m = 4$. **(b).** $p = 3, r = 7/8, m = 7$. **(c).** $p = 5, r = 5/4, m = 4$.

5. Find the 7-adic expansion of each of: 200 and 3/14. Determine the member of $\mathbb{Q}$ expressed by the 5-adic expansion $2, \overline{34}$.

6. Let $x \in \mathbb{Q}$. Show that $x \in \mathbb{Z} \iff \big(x \in \mathbb{Z}_p \text{ for all } p\big)$.

7. Decide whether there exists $x \in \mathbb{Q}_p$ such that $x^2 = -28$ for each of: $p = 2, 3, 5, 7, 11$.

8. Show that $(X^2 - 2)(X^2 - 17)(X^2 - 34)$ has a root in $\mathbb{R}$ and in every $\mathbb{Q}_p$, but not in $\mathbb{Q}$.

9. Is 4 a cube in $\mathbb{Q}_3$? Is 28 a cube in $\mathbb{Q}_3$? Is 13 a cube in $\mathbb{Q}_7$?