Lie Groups

Section C course Hilary 2021

kirwan@maths.ox.ac.uk

Example sheet 3

Question 1. Let $\mathfrak{sl}(2,\mathbb{R})$ denote the space of 2×2 real matrices of trace zero.
Show that $\mathfrak{sl}(2,\mathbb{R})$ is a Lie algebra with basis

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

and work out the bracket relations for e, f, h.
By considering subalgebras of this Lie algebra, show that it is not isomorphic to $\mathfrak{su}(2)$.

Question 2. Let $\varphi : G_1 \to G_2$ be a Lie group homomorphism. Show that

$$\ker \varphi \subset G_1$$

is a closed (hence embedded) Lie subgroup with Lie algebra

$$\ker(D_1\varphi) \subset \mathfrak{g}_1.$$

*A vector subspace $J \subset (V, [\cdot,\cdot])$ of a Lie algebra is called an **ideal** if*

$$[v, j] \in J \text{ for all } v \in V, j \in J.$$

Show that ideals are Lie subalgebras. Show that for a Lie subgroup $H \subset G$, with H, G connected,

$$\boxed{H \subset G \text{ is a normal subgroup} \Leftrightarrow \mathfrak{h} \subset \mathfrak{g} \text{ is an ideal}}$$

(You may find it helpful to first show the identity $ge^Y g^{-1} = e^{\mathrm{Ad}\,(g).Y}$ for $g \in G$ and $Y \in \mathfrak{g}$).
*The **centre** of a Lie algebra $(V, [\cdot,\cdot])$ is*

$$Z(V) = \{v \in V : [v, w] = 0 \text{ for all } w \in V\}.$$

For G connected, prove that the centre of the group G is[1]

$$\boxed{Z(G) = \ker(\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g}))}$$

Deduce that the centre of G is a closed (hence embedded) Lie subgroup of G which is abelian, normal and has Lie algebra

$$\mathrm{Lie}(Z(G)) = Z(\mathfrak{g}).$$

Finally deduce that, for G connected,

$$\boxed{G \text{ is abelian} \Leftrightarrow \mathfrak{g} \text{ is abelian.}}$$

Question 3. Show that

$$[X, Y] = 0 \Rightarrow \exp(X + Y) = \exp(X)\exp(Y).$$

Prove that if G is a connected Lie group with $Z(G) = \{1\}$ then G can be identified with a Lie subgroup of $GL(m,\mathbb{R})$, for some m, so $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{gl}(m,\mathbb{R})$.

If $(V, [\cdot,\cdot])$ is a Lie algebra with $Z(V) = \{0\}$, show that V is the Lie algebra of some Lie group.

Question 4. Find all the connected Lie subgroups of $SO(3)$.
Hint. Use the results from Q.3 of Question sheet 2.

[1] Recall the centre of a group is $Z(G) = \{g \in G : hg = gh \text{ for all } h \in G\} = \{g \in G : hgh^{-1} = g \text{ for all } h \in G\}$.

Question 5. Show that Lebesgue measure $\mathbf{dx}$ is the bi-invariant Haar measure on $\mathbb{R}^n$ viewed as an additive group.
Find the bi-invariant Haar measure on $(\mathbb{R}_{>0}, \times)$, the multiplicative group of positive reals,

Question 6. Give an example of an *irreducible* representation of S^1 on $\mathbb{R}^2$. Describe what happens to this representation when we complexify it.