Lie Groups

Section C course Hilary 2021

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Example sheet 4

1. Check the following properties hold for a character χ_V associated to a representation V of a compact Lie group G.

1. $\chi_V(1) = \dim V$
2. χ_V is invariant under conjugation, $\chi_V(hgh^{-1}) = \chi_V(g)$
3. $\chi_V = \chi_W$ for equivalent reps $V \simeq W$
4. $\chi_{V \oplus W}(g) = \chi_V(g) + \chi_W(g)$
5. $\chi_{V \otimes W}(g) = \chi_V(g) \cdot \chi_W(g)$
6. $\chi_{V^*}(g) = \chi_V(g^{-1}) = \overline{\chi_V(g)}$

2. Which of the irreducible representations V_n of $SU(2)$ may be regarded as representations of $SO(3)$?

Recalling which of the V_n have a real structure, deduce that for each natural number n we have a real $(2n+1)$-dimensional representation W_n of $SO(3)$.

Show further that the character of W_n is given by

$$\sum_{k=0}^{2n} e^{i(n-k)t}.$$

3. Show that a maximal torus in a compact Lie group is maximal among connected Abelian subgroups.

4. Find the Weyl group of the unitary group $U(n)$.

5. Let B denote the subgroup of $GL(3, \mathbb{C})$ consisting of invertible matrices of the form

$$\begin{pmatrix} \alpha & a & b \\ 0 & \beta & c \\ 0 & 0 & \gamma \end{pmatrix} \quad :$$

Check that B is indeed a subgroup, and that there is a homomorphism ϕ from B onto the *complex torus* $T_{\mathbb{C}} \cong (\mathbb{C}^*)^3$ of diagonal elements of B. Show $\ker \phi$ may be identified with the subgroup U consisting of elements of B with diagonal entries equal to 1.

Show further that the elements of U with $a = c = 0$ form a normal subgroup of U.

What are the maximal compact connected subgroups of T, B and U? (no need to give detailed proofs).