Geometric Group Theory

Problem Sheet 1

1. Show that the free group of rank r, F_r, has exactly $2^r - 1$ subgroups of index 2. (*hint: consider homomorphisms to* $\mathbb{Z}_2$).

2. i. Show that F_2 has a free subgroup of rank 3.
ii. Show that F_2 has an infinite index free subgroup of rank 2.
iii. Show that F_2 has a free subgroup of infinite rank.

3. Prove the ping-pong lemma: Let G be a group acting on a set S and let $a, b \in G$ be infinite order elements. If there are non empty disjoint subsets A, B of S such that $a^n B \subseteq A$, $b^n A \subseteq B$ for all $n \in \mathbb{Z} \setminus \{0\}$ then $\{a, b\}$ generate a free subgroup of rank 2 of G. (*hint: if* $w = a^{k_1} b^{k_2} ... a^{k_n}$ *then show that* $wB \subseteq A$. *Otherwise replace* w *by a conjugate and use the same argument*).

4. Show that the matrices

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

generate a free subgroup of $SL_2(\mathbb{Z})$. (*hint: apply the ping pong lemma to the usual action of* $SL_2(\mathbb{Z})$ *on* $\mathbb{R}^2$ *with* $A = \{(x, y) : |x| > |y|\}$, $B = \{(x, y) : |x| < |y|\}$).

5. i) Let $G_1 = \langle S_1 | R_1 \rangle$, $G_2 = \langle S_2 | R_2 \rangle$. Find a presentation for the direct product $G_1 \times G_2$.
ii) If $G = \langle S | R \rangle$ find a presentation for the abelianization of G.

6. Show that the group $G = \langle a, b | ababa = 1 \rangle$ is abelian.

7. i. Show that the group

$$G = < x, y | x^2 = y^3 >$$

is not trivial.
ii. Show that the group G is isomorphic to the group $H = < a, b | aba = bab >$

8. Show that every finitely presented group has a finite presentation in which every relation is a word of length at most 3.