C2.6 Introduction to Schemes

Prof. Alexander F. Ritter
University of Oxford
HT 2020
ritter@maths.ox.ac.uk

Feedback and corrections are welcome!

EXERCISE SHEET 4

1) i) $F, G \subseteq H$ subsheaves then $F = G \iff F_x = G_x \;\; \forall x \in X$ (top. space)

ii) $F \xrightarrow{\varphi} G$ in $Ab(X) \Rightarrow \operatorname{Ker}\varphi : U \longmapsto \operatorname{Ker}(\varphi_U)$ is a sheaf (whereas for $\operatorname{Coker}\varphi$ and $\operatorname{Im}\varphi$ we must sheafify)

iii) Also show $(\operatorname{Ker}\varphi)_x = \operatorname{Ker}(\varphi_x)$ and $(\operatorname{Im}\varphi)_x = \operatorname{Im}(\varphi_x)$ (recall the definition $\operatorname{Im}\varphi = \operatorname{Ker}(G \to \operatorname{Coker}\varphi)$)

Show φ injective $\iff \varphi_x$ injective $\forall x$, φ surjective $\iff \varphi_x$ surjective $\forall x$

Deduce that $F \to G \to H$ in $Ab(X)$ exact $\iff$ $F_x \to G_x \to H_x$ exact $\forall x$

iv) $F \xrightarrow{\varphi} G$ with φ_x surjective $\Rightarrow \forall\, s \in G(U), x \in U$, $\exists$ open $x \in V \subseteq U$, $\exists\, t \in F(V)$ with $\varphi(t) = s|_V$

v) "surjectivity means local liftability" $F \xrightarrow{\varphi} G$ surj $\iff \forall s \in G(U)$, $\exists$ open cover $U = \cup U_i$ $\exists\, t_i \in F(U_i)$ with $F(t_i) = s|_{U_i}$

vi) $X = \mathbb{C} \setminus \{\frac{1}{n} : n \geq 1 \in \mathbb{N}\}$ $\mathcal{O}_X(U) := \{\text{holomorphic functions } U \to \mathbb{C}\}$ (holomorphic = complex-differentiable)

Euclidean topology $\mathcal{O}_X^*(U) := \{\text{nowhere zero holomorphic functions } U \to \mathbb{C}\}$

Show $\exp : \mathcal{O}_X \to \mathcal{O}_X^*$ is surjective (for $f \in \mathcal{O}_X(U)$, $\exp(f) \in \mathcal{O}_X^*(U)$ is the complex exponential)

but $\exp_U : \mathcal{O}_X(U) \to \mathcal{O}_X^*(U)$ not surjective no matter how small the open $0 \in U$.

vii) $F \xrightarrow{\varphi} G$ hom of $\mathcal{O}_X$-mods ($(X, \mathcal{O}_X)$ ringed space), G finite type then: φ_x surj. (some $x \in X$) $\Rightarrow \varphi_U : F|_U \to G|_U$ surj. some open $x \in U$

(In notes 6.3 we used this: we had $\mathcal{O}_{X,x}^{\oplus n} \xrightarrow{\cong} F_x$, and assuming F finite type we claimed that $\mathcal{O}_U^{\oplus n} \to F|_U$ surjective on some open $x \in U$).

viii) $F \xrightarrow{\varphi} G$ hom of $\mathcal{O}_X$-mods, G coherent, F finite type then: φ_x inj. $\Rightarrow \varphi_U : F|_U \to G|_U$ inj. some open $x \in U$

(Hint First check $\ker\varphi$ finite type (doesn't use φ_x inj) by considering $\ker(\mathcal{O}|_U^{\oplus n} \twoheadrightarrow F|_U \xrightarrow{\varphi_U} G|_U)$)

2) Motivation: why is Nakayama's Lemma useful in geometry?

"Transferring information from pointwise to infinitesimal to local":

Recall Nakayama's Lemma: (there are many versions of this, the proofs are very similar)

$\boxed{\text{$R$ ring, $p \in \operatorname{Spec} R$, M f.g. R-mod. If $n_1, \ldots, n_d \in M$ is a basis for the $k(p)$-vector space $M_p \otimes_{R_p} k(p)$ ($\cong M_p/p \cdot M_p$) then $n_1, \ldots, n_d$ generate the R_f-module M_f for some $f \in R \setminus p$ (indeed it is a minimal generating set)}}$

(when R is a local ring with max ideal m, this becomes: $M/mM = \langle n_1, \ldots, n_d \rangle \Rightarrow M = \langle n_1, \ldots, n_d \rangle$)

i) $(X, \mathcal{O}_X)$ scheme, $F \in QCoh(X)$ finite type then call $F(x) = F_x \otimes_{\mathcal{O}_{X,x}} k(x)$ the fiber

Given $s_1, \ldots, s_n \in F(U)$ on open $x \in U$, if $(s_1)_x, \ldots, (s_n)_x$ generate the fiber then possibly after shrinking U, show the $s_1, \ldots, s_n$ also generate $F|_U$.

Deduce: • if $F(x) = 0$ then $F|_U = 0$ some open $x \in U$.

• $x \longmapsto \dim F(x)$ is upper-semicontinuous, i.e. $\{\dim < d\} \subseteq X$ is open (since integer-valued can also take $\leq d$.)

Algebra fact in R-mods: $0 \to M_1 \to M_2 \to M_3 \to 0$ exact, M_2 flat (e.g. free) then M_3 flat $\iff M_1/IM_1 \to M_2/IM_2$ injective $\forall$ ideal $I \subseteq R$.

ii) Let $F \in \mathcal{O}_X$-Mods be locally finitely presented (e.g. $F \in Coh(X)$). Prove: $\boxed{F \in Vect(X) \iff F \text{ flat } \mathcal{O}_X\text{-mod}}$

(Hints. Rewrite the algebra fact in case R local ring, check it's enough to consider I = the max ideal. The key is to reach an exact sequence of type $0 \to N_x/m_x N_x \to k(x)^{\oplus n} \to F_x/m_x F_x \to 0$ & use (i))

3) Motivation: $\mathrm{Vect}(\operatorname{Spec} R) \longleftrightarrow \widetilde{M}$ for f.g. projective R-mods M

$X = \operatorname{Spec} R$, M R-mod.

Consider the conditions

[You may need: projective R-mod $\iff$ direct summand of free R-mod
also: P projective $\iff$ every exact $0 \to M \to N \to P \to 0$ splits

⓪ $\widetilde{M} \in \mathrm{Vect}(X)$ i.e. locally free of finite rank, i.e. $\exists$ cover $X = \cup D_{f_i}$ with M_{f_i} f.g. free R_{f_i}-mod

① M finitely presented and flat

② M finitely presented and $M_{\mathfrak{m}}$ free $R_{\mathfrak{m}}$-mod $\forall$ max ideal $\mathfrak{m}$ (can also use all prime ideals)

③ M is the direct summand of a finite rank free module

④ M f.g. projective

i) Prove ⓪ $\iff$ ① Hints for $\Rightarrow$ use ex. 2(iii), for $\Leftarrow$ compare the proof in Sec. 3.1 of notes, use tricks from Sec. 3.0, and use fact $\left.\begin{array}{l} 0 \to K \to M_1 \to M_2 \to 0 \text{ exact} \\ M_1 \text{ f.g.}, M_2 \text{ finitely presented} \end{array}\right\} \Rightarrow K$ f.g.

ii) Prove ⓪&① $\iff$ ② and ④ $\iff$ ③ $\Rightarrow$ ①

iii) Finally prove ⓪ $\Rightarrow$ ④ (Hint use fact about localization: $S^{-1}\mathrm{Hom}_R(M,N) = \mathrm{Hom}_{S^{-1}R}(S^{-1}M, S^{-1}N)$ for M finitely presented & use Sec. 3.0 of notes)

4) i) $X = \operatorname{Spec} R$, M R-mod

- Show that $\mathcal{L} = \widetilde{M}$ is line bundle $\iff \forall p \in X, \exists f \in R \setminus p : M_f \cong R_f$
- $\mathcal{L} = \widetilde{M}$ line bundle $\iff M$ f.g. projective R-mod with $\dim_{K(p)} M \otimes K(p) = 1 \quad \forall p \in X$.

Deduce that every line bundle on $\mathbb{A}^1_k = \operatorname{Spec} k[t]$ is trivial (k field) ← Hint structure theorem for f.g. mods over PID

ii) Let $F \in \mathrm{Vect}(X)$ (any scheme). Describe the transition function (in terms of transition functions of F) of the dual $F^\vee := \mathrm{Hom}_{\mathcal{O}_X}(F, \mathcal{O}_X)$.
Deduce that for a line bundle $\mathcal{L}$, the transition function of $\mathcal{L}^\vee$ is the inverse of that of $\mathcal{L}$ and $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L}^\vee = \mathcal{L} \otimes_{\mathcal{O}_X} \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X) \xrightarrow{\cong} \mathcal{O}_X$ via the natural evaluation map.

iii) Let M, N be R-mods. Suppose $M \otimes_R N \underset{\cong}{\xrightarrow{\varphi}} R$. Pick $m_i \in M, n_i \in N$ with $\varphi(\sum_{i=1}^d m_i \otimes n_i) = 1$.
Check that $M \to M$, $m \longmapsto \sum \varphi(m \otimes n_i) m_i$ is an isomorphism which factorizes as $M \to R^d \to M$, and deduce that M is a summand of R^d.

iv) $\mathcal{L} \in \mathcal{O}_X$-Mods is line bundle $\iff \exists F \in \mathrm{QCoh}(X) : F \otimes_{\mathcal{O}_X} \mathcal{L} \cong \mathcal{O}_X$ (Def. $\mathcal{L}$ being invertible sheaf)
(Hint combine (iii) with ex. 3③)

5) FACT every line bundle on $\mathbb{A}^n_k$ is trivial

i) Calculate $\mathrm{Pic}(\mathbb{P}^n) = \{$isomorphism classes of line bundles on $\mathbb{P}^n\}$ with group operation $\otimes_{\mathcal{O}_{\mathbb{P}^n}}$.
Indeed show it is $\cong \mathbb{Z}$, generated by $\mathcal{O}(1)$ (defined in the notes)

ii) Compute $\Gamma(\mathbb{P}^n, \mathcal{O}(d))$ for $d \in \mathbb{Z}$ $\quad (\mathcal{O}(d) = \mathcal{O}(1)^{\otimes d})$

iii) Let p be the point $(x) \in \operatorname{Spec} k[x] = A_0$ in $\mathbb{P}^1_k = A_0 \cup A_1$. Show that $\mathcal{O}(-1) \cong$ ideal sheaf of $\{p\}$
Let Z be the closed subscheme $\operatorname{Spec}(k[x]/x^d) \subseteq A_0 \subseteq \mathbb{P}^1_k$. Show $\mathcal{O}(-d) \cong$ ideal sheaf of Z
what is the ideal sheaf of d closed points $\{p_1, \dots, p_d\} \subseteq \mathbb{P}^1$?

iv) OPTIONAL If two graded R-mods M, N over graded ring R satisfy $M_n \cong N_n \ \forall n \geq d \Rightarrow \widetilde{M} = \widetilde{N}$ see Sec. 10 notes

6) i) $\mathcal{C}$ = abelian cat. Show that if every object $M \in \mathcal{C}$ has an injective morph $M \to I$ into an injective object, then every object M admits an injective resolution. (We say $\mathcal{C}$ has "enough injectives")

FACT Cat. Ab of abelian groups has enough injectives

ii) $F \in \mathrm{Ab}(X)$ (sheaves; X top. space). Pick $I_x \in \mathrm{Ab}$ s.t. $F_x \to I_x$ injective morph and I_x injective object in Ab
Show that $I := \prod_{x \in X} (\varphi_x)_* I_x \in \mathrm{Ab}(X)$ is an injective object admitting an inj. morph $F \to I$ (hence $\mathrm{Ab}(X)$ has enough injectives)
inclusion map $\varphi_x : \{x\} \hookrightarrow X$ of a point.