Solutions to Sheet 1

Rolf Suabedissen

February 10, 2020

Health Warning: Although I have set the problem sheet, these solutions are still not 'model' solutions. Almost every year I discover something new or worthwhile to say. Some of these new things are different approaches from students which I find interesting, but some of them are also subtle gaps in the current solutions. Really, you should use a Theorem Checker/Prover (e.g. Isabelle) to prove these to get rid of all the subtle gaps.

Also, this is very long and a little rambling. You can write much shorter, essentially perfect solutions.

Question 1

Some general points:

- The point of this question is to understand that the $\in$-relation is 'just' a binary relation. One way to understand it is to draw directed graphs (with the $a \leftarrow b$ standing for $a \in b$) and play around with it.

- Try to understand what a particular $\in$-formula means in the given structure.

  For example in a total order $<$ the formula $z \subseteq x \equiv \forall t\ (t \in z \to t \in x)$ translates to $\forall t\ (t < z \to t < x)$ which is equivalent to $z \leq x$.

  Similarly, the emptyset formula $z = \emptyset \equiv \forall t (t \in z \leftarrow \text{False})$ corresponds to $\forall t (t < z \leftarrow \text{False})$, i.e. that z is minimal.

- If you try to understand quantified formulae 'relativized' to $(\mathbb{Q}, <)$ there are two different $\in$, the external one ($x \in \mathbb{Q}$ or more generally $x \in C$ for some class C, i.e. a formula with one free variable) and the internal one $\in^{\mathbb{Q}}$ which is $<$. Thus going back to $\phi(z, x) \equiv \forall t\ (t \in z \to t \in x)$ we get $\phi^{(\mathbb{Q},<)}(z, x) \equiv \forall_{\mathbb{Q}} t\ (t < z \to t < x)$ where $\forall_{\mathbb{Q}} t$ quantifies over all rational t and is usually written as $\forall t \in \mathbb{Q}$.

- Different authors use different (but similar) axioms for **ZFC**. In the context of all the other axioms these are (usually) equivalent, but changing more than one axiom can lead to unexpected consequences.

Extensionality : Suppose $q, p \in \mathbb{Q}$ with $q \neq p$, wlog $q < p$. Then there is $r \in (q,p) \cap \mathbb{Q}$ and $r < p$ and $\neg(r < q)$. Hence $\mathbb{Q} \models$ **Extensionality**

Emptyset : If $q \in \mathbb{Q}$ then $q - 1 \in \mathbb{Q}$ and $q - 1 < q$. Thus $\neg\mathbb{Q} \models$ **Emptyset**. Note that this implies that $\mathbb{Q}$ does not satisfy **Separation** either since $\mathbb{Q}$ is non-empty.

Powerset: First note that $r \subseteq q$ means $\forall t\ (t < r \rightarrow t < q)$, i.e. $r \leq q$.

Here is a subtlety: if you take the weak **Powerset** axiom:

$$\forall x \exists z \forall r\ \ [r \in z \leftarrow r \subseteq x]$$

then $\mathbb{Q} \models$ **Powerset**: Let $q \in \mathbb{Q}$. Try any $z \geq q$: if $r \in \mathbb{Q}$ such that $r \subseteq q$ which means $r \leq x$ then certainly $r \leq q + 1$.

However if you take the strong **Powerset** axiom:

$$\forall x \exists z \forall r\ \ [t \in z \leftrightarrow r \subseteq x]$$

then $\mathbb{Q} \not\models$ **Powerset**: take $q = 0$ and any $z \in \mathbb{Q}$: if $q < z$ then take $r \in \mathbb{Q}$ with $q < r < z$ (e.g. $r = (q+z)/2$) and note that $r \not\leq q$ but $r < z$. If $z \leq q$ then take $r = q$ and note that $r \leq q$ but $r \not< z$.

So in the absence of **Separation** (see below) the distinction becomes important.

Infinity: As stated, $\neg\mathbb{Q} \models$ **Infinity** since there is no $y \in \mathbb{Q}$ such that $\forall z\ z \notin y$. (Exercise: what is a successor in a linear order and when does a linear order have a 'non-empty' 'inductive' point.) Also for every rational q and any n (in the meta-theory) we have $q - 1, \ldots, q - n < q$ so q is not 'finite'.

Separation: Let $\phi(r) \equiv r < r$ and fix $q \in \mathbb{Q}$. We ask whether there is any $p \in \mathbb{Q}$ such that $\forall t\ \ [t < p \leftrightarrow t < q \wedge \phi(t)]$? Suppose there was some such p and consider $t = \min\{p-1, q-1\} \in \mathbb{Q}$. Then $t < p$ but $\neg\phi(t)$ a contradiction.

Let $\phi(r, q) \equiv r < q$. Fix $q, s \in \mathbb{Q}$. We ask whether there is any $p \in \mathbb{Q}$ such that $\forall t\ \ [t < p \leftrightarrow t < s \wedge \phi(t, q)]$. Clearly $p = \min\{s, q\}$ satisfies this.

Note that you cannot reference any particular element of $\mathbb{Q}$ without using a parameter: $\phi(v) \equiv t = 0$ is not a valid formula (since our language does not contain any constants). You can of course use $\phi(v_1, v_2) \equiv v_2 = v_1$ and then consider the parameter $a_1 = 0$ and $x = 1$ to form $z = \{t \in 1 : \phi(0, t)\}$ which is shorthand for $t < z \leftrightarrow (t < 1 \wedge t = 0)$ and it is clear that no such z exists (in $(\mathbb{Q}, <)$).

Question 2

By recursion on ω: we need to show that $P(x) = x \cup \{\{y, z\} : y, z \in x\}$ is a set (see below) and use $\{\langle x, P(x)\rangle : x \in U\}$ as our class function F to obtain M_n for $n \in \omega$ such that $M_0 = \{\emptyset\}$ and $M_{n+1} = P(M_n)$. We finally use **Replacement**,

Infinity (to get that ω is a set) and **Union** to define $M = \bigcup \{M_n : n \in \omega\}$ as a set.

Note that $P(\emptyset) = \emptyset$, so we have to start with $\{\emptyset\}$. Of course, we could use $\hat{P}(x) = x \cup \{y \in \mathcal{P}(x) : \exists t_1, t_2 \; y \subseteq \{t_1, t_2\}\}$ and start with $M_0 = \emptyset$.

First we show by induction on n that no element of M_n contains more than two elements (straightforward) and deduce that no element of M contains more than two elements. For transitivity assume $x \in M$. Find the least n such that $x \in M_n$. If $n = 0$ then $x = \emptyset$ and we are vacuously done. Otherwise $n = m + 1$ for some m and there are $y, z \in M_n$ with $x = \{y, z\}$. So now assume $t \in x$. Then $t = y$ or $t = z$. In either case $t \in M_n \subseteq M$ as required.

Note that clearly $M_n \subseteq M_{n+1}$ by construction and hence by induction we have $n \leq m$ implies $M_n \subseteq M_m$. Thus if $x, y \in M$ then $\{x, y\} \in M$ (for a more formal proof see below).

For the axioms (key point: after getting your 'candidate' you have to check that it belongs to M and that M believes the right stuff about it!).

- **Extensionality** follows from transitivity of M.

- **Emptyset** is trivial, but remember that you need to remark that being the emptyset is absolute for $M, U = \{x : x = x\}$ (because they are transitive) so $\emptyset^U = \emptyset^M$.

- **Pairing** follows by construction: let $x, y \in M$ and find n, m such that $x \in M_n, y \in M_m$. Wlog $n \leq m$ and by the note above we then have $x, y \in M_m$ so that $z = \{x, y\} \in M_{m+1} \subseteq M$. Because $z = \{x, y\}$ is absolute (for transitive non-empty classes) and M is non-empty and transitive, we have $[z = \{x, y\}]^M$ as required.

- For **Separation**, let $\phi(y; v_1, \ldots, v_n)$ be a formula, $a_1, \ldots, a_n \in M$ and $u \in M$. Let n be least such that $u \in M_n$. If $n = 0$ then $u = \emptyset$, so let $z = \emptyset \in M$ and $M \models t \in z \leftrightarrow t \in u \wedge \phi(t; a_1, \ldots, a_n)$ is vacuously true. So assume $n = m + 1$ for some m. By leastness, there are $x, y \in M_m$ with $u = \{x, y\}$. Set $z = \left\{t \in u : \phi(t; a_1, \ldots, a_n)^M\right\}$. Then z is one of $\emptyset$, $\{x\}$, $\{y\}$ or $\{x, y\}$ all of which belong to M. Finally $M \models t \in z \leftrightarrow t \in u \wedge \phi(t; a_1, \ldots, a_n)^M$ as required.

- For **Replacement** assume that $\phi(x, y)$ is a formula, $d \in M$ and

  $$\forall x \in d \forall y, y' \in M \left[\phi(x, y)^M \wedge \phi(x, y')^M \rightarrow y = y'\right].$$

  (This is equivalent to the relativization of M believes that ϕ codes a function on d).

  Define $z = \left\{y \in M : \exists x \in M (x \in s \wedge \phi(x, y)^M)\right\}$. Firstly, z is a set by **Separation**. If we can show that $z \in M$ then we are done (because by construction it is the 'right' set). So, let n be least such that $s \in M_n$. If $n = 0$ then $s = \emptyset$ and hence $z = \emptyset$ and clearly $M \models \forall t \; [t \in z \leftrightarrow t \in s \wedge \ldots]$. Otherwise let $n = m + 1$ and by leastness find $u, v \in M_m$ with $s = \{u, v\}$. Then there are is at most one $a \in M$ with $\phi(u, a)^M$ and only one $b \in M$

with $\phi(v,b)^M$. But all of $\emptyset, \{a\}, \{b\}, \{a,b\}$ (depending on whether or not $a, b \in M$ exist) belong to M. So $z \in M$ as required.

- We claim that **Powerset** fails in M: Note that $\subseteq$ is absolute for M, U by transitivity, so we don't specify which one we mean.

  Let $x, y \in M$ be distinct (e.g. $x = \emptyset$, $y = 1 = \{\emptyset\}$) and set $t = \{x, y\} \in M$. Suppose that there is $z \in M$ such that $\forall s \in M\, [s \subseteq t \rightarrow s \in z]$ (this is the 'weaker' version of **Powerset** relativized to M noting that $\subseteq$ is absolute for M).

  We then have: $\emptyset \in M$ and of course $\emptyset \subseteq s$; $\{x\}, \{y\} \in M$ and $\{x\}, \{y\} \subseteq s$; $\{x, y\} \in M$ and $\{x, y\} \subseteq s$. Thus $\emptyset, \{x\}, \{y\}, \{x, y\} \in z$ and so z has at least four distinct elements, contradicting $z \in M$.

- Similarly, **Union** fails in M: take $x, y, r, t \in M$ distinct (it is not difficult to write down four distinct elements of M) and form $a = \{x, y\}, b = \{r, t\}, c = \{a, b\} \in M$. As in the argument for **Powerset**, if there is $z \in M$ with $M \models z = \bigcup c$ then $x, y, r, t \in c$ (because $z = \bigcup c$ is absolute) leading to a contradiction.

- Finally, **Infinity**: this is tricky since $\alpha + 1 = \alpha \cup \{\alpha\}$ might not make sense (as **Union** does not hold). So we have to go back to the $\in$-definitions of these concepts, i.e.

  $$z = \alpha + 1 \equiv \alpha \in z \wedge [\forall t \in \alpha\ t \in z] \wedge \forall t \in z\ \ [t \in \alpha \vee t = \alpha]$$

  and $\alpha + 1 \in w \equiv \exists z \in w\ z = \alpha + 1$.

  Now assume that **Infinity** holds as witnessed by w. Then $\emptyset \in w$, and $M \models \{\emptyset\} = \emptyset + 1$ and $M \models \{\emptyset, \{\emptyset\}\} = (\emptyset + 1) + 1$ noting that all these are in M. Write $0, 1, 2$ for these three respectively. Now we need to check whether $M \models \exists z \in w\ z = 2 + 1$. As before we would require $0, 1, 2 \in z$ contradicting $z \in M$.

- Finally **Choice**: There are of course multiple versions of the Choice axiom (which are equivalent under ZF). We will look at two of them and show that $M \models$ **Choice**.

  Choice: For every set u of disjoint non-empty sets there is a transversal v, i.e. there is v such that for every $y \in u$, $v \cap y$ is a singleton. Formally:

  $$\forall u\, [[\forall y \in x\ y \neq \emptyset] \rightarrow \exists v\ \ [\forall y \in u\ \exists t\ v \cap y = \{t\}]]$$

  Note that by transitivity of M, we have $(x \cap y = \emptyset)^M$ if and only if $x \cap y = \emptyset$ and $(x = \emptyset)^M$ if and only if $x = \emptyset$.

  So let $u \in M$. If $u = \emptyset$ then $v = \emptyset$ vacuously works. Otherwise, let $u = \{x, y\}$ (as above). By transitivity and the above we may assume (the first line relativized to M is equivalent to it not relativized to M) that u consists of pairwise disjoint non-empty sets, i.e. $x \neq \emptyset$, $y \neq \emptyset$ and

$x \cap y = \emptyset$. As $x, y \neq \emptyset$, there are $a, b \in M$ with $x = \{a, b\}$ and distinct (from a, b but possibly not from each other) $c, d \in M$ with $y = \{c, d\}$. So let $v = \{a, c\} \in M$ and observe that this v works. (We have not used choice here - finitely many 'choices' are covered by logic and induction!)

Well ordering principle: For every set u there is a well-ordering $\leq$ on u, formally:

$$\forall u \exists R \ [R \text{ is a well-order on } u]$$

where of course R is a well-order on u means

$$\forall t \in R \, [t \text{ is a 2-tuple}] \wedge \quad (1)$$

$$\forall t \in u \, \langle t, t \rangle \notin R \wedge \quad (2)$$

$$\forall t, r, s \in u \, [\langle t, r \rangle \in R \wedge \langle r, s \rangle \in R \to \langle t, s \rangle \in R] \wedge \quad (3)$$

$$\forall y \ \ [y \subseteq u \wedge y \neq \emptyset \to \exists m \in y \forall m' \in y \ \ \langle m, m' \rangle \in R] \quad (4)$$

So assume that $u \in M$. If $u = \emptyset$ or $u = \{x\}$ for some $x \in M$, then $R = \emptyset \in M$ is a well-order on u (check the condidtions). Otherwise $u = \{x, y\}$ for $x, y \in M$ with $x \neq y$ and we define $R = \langle x, y \rangle = \{\{x\}, \{x, y\}\}$. Note that $R \in M$ (it is a pair of pairs of elements of M) and it is straightforward to check that $M \models R$ is a well-order on u.

Question 3

Let a be non-empty, transitive and let m be its $\in$-minimal element (from Foundation). If $x \in m$ then by transitivity $x \in a$, contradicting minimality of m. So $m = \emptyset$ as required.

Please avoid trying to assume that $\emptyset \notin a$ and defining a decreasing infinite $\in$-chain (which would contradict **Foundation**). This will most likely require (some form of) **Choice** and would be messier than necessary.

Question 4

Suppose x, y are sets. Write $0 = \emptyset$ and $1 = \{\emptyset\} = \mathcal{P}(\emptyset)$. Then $0, 1 \subseteq 1$ so $0, 1 \in \mathcal{P}(1)$ so by **Separation** $\{0, 1\}$ is a set (in fact, $\mathcal{P}(1) = \{0, 1\}$ so another application of **Powerset** can avoide **Separation**). By Replacement (with $\phi(v_1, v_2, r, t)$ as

$$(r = 0 \wedge t = v_1) \vee (r = 1 \wedge t = v_2)$$

and parameters $v_1 = x, v_2 = y$ and $d = \{0, 1\}$) this gives that $\{x, y\}$ is a set.

Note that we are using a very weak form of **Replacement** here.

Also note that it is not 'clean' to say something like: apply **Replacement** with $\phi(r, t) \equiv (r = 0 \wedge t = x) \vee (r = 1 \wedge t = y)$. This appears to define one formula for every 'instance' of **Pairing**. Also, the above is not a formula of LST (if you think of x, y as constants).

Question 5

Clearly being well-ordered implies being totally ordered so (i) implies (ii). We focus on (ii) implies (i): Suppose that α is transitive and totally ordered by $\in$. Let $x \subseteq \alpha$ and assume that $x \neq \emptyset$. Apply **Foundation** to find $m \in x$ such that $m \cap x = \emptyset$. Since α is transitive, $m \in \alpha$ and by construction m is the $\in$-minimal element of x.

For the deduce, note that α is transitive and totally ordered by $\in$ is a Δ_0 formula, so absolute for transitive non-empty classes $A \subseteq B$. As long as A, B satisfy **Foundation**, the above show that $A \models \alpha$ is transitive and well-ordered by $\in$ if and only if $A \models \alpha$ is transitive and totally-ordered by $\in$ if and only if $B \models \alpha$ is transitive and totally-ordered by $\in$ if and only if $B \models \alpha$ is transitive and well-ordered by $\in$, as required.

Question 6

The most difficult part is to find out what you are actually asked to do. We want to show that: If

- A, B satisfy enough of ZF so that the Recursion Theorem on On holds and
- $a \in A$ and
- that F is a formula such that $A \models F$ is a class function (we will write $F^A(a)$ for the unique $y \in A$ with $A \models F(a, y)$) and
- $B \models F$ is a class function (similarly $F^B(b)$ is the unique $y \in B$ such that $B \models F(a, b)$) and
- F is absolute for A, B (i.e. for $a \in A$, $F^A(a) = F^B(a)$) and
- G_A (resp. G_B) are formulae such that

$$A \models G \text{ is a class function on } On^A \wedge \tag{5}$$
$$G(0^A) = a \tag{6}$$
$$\wedge \forall \alpha \in \mathrm{On}^A \ G_A((\alpha+1)^A) = F^A(G_A(\alpha)) \tag{7}$$
$$\wedge \forall \gamma \in \mathrm{Lim}^A \ G_A(\gamma) = \left(\bigcup \{G_A(\beta) : \beta \in \alpha\}^A\right)^A \tag{8}$$

  (resp. the above for B and G_B) where all the superscript As mean that we should interpret this formula in A

then
$$\forall \alpha \in On^A \ G_A(\alpha) = G_B(\alpha).$$

For the proof, we first note that since A, B are non-empty transitive classes satisfying enough of ZF we have $\emptyset^A = \emptyset^B$, $On^A \subseteq On^B$ and $Lim^A \subseteq Lim^B$

(being an ordinal is absolute and being a successor ordinal is absolute, hence being a limit ordinal is absolute).

So assume there is $\alpha \in On^A$ with $G_A(\alpha) \neq G_B(\alpha)$. Since A satisfies enough of ZF, there is a minimal such α, say α_0.

Case $\alpha_0 = 0^A = 0^B$: Then $G_A(\alpha_0) = a = B_B(\alpha_0)$, a contradiction.

Case α_0 is a successor (in A): Being a successor is absolute for A, B, so α_0 is successor in B. Let $\beta_A \in On^A$ be such that $A \models \alpha_0 = \beta_A + 1$ and similarly for β_B. Since **Pairing** and **Union** are absolute, $A, B \models \beta_A + 1 = \beta_B + 1$ and it follows that $A, B \models \beta_A = \beta_B$. We will simply write β for β_A. Since $\beta \in \alpha_0$, by minimality of α_0 we must have

$$G_A(\alpha_0) = F^A(G_A(\beta)) = F^A(G_B(\beta) = F^B(G_B(\beta)) = G_B(\alpha_0)$$

(where the second $=$ comes from the minimality of α_0 and the third from absoluteness of F), giving another contradiction.

Case α_0 is a limit (in A): Again, α_0 will be a limit in B. Now apply minimality of α_0 to see that for $\beta \in \alpha_0$, $G_A(\beta) = G_B(\beta)$, so that $\{G_A(\beta) : \beta \in \alpha\}^A = \{G_B(\beta) : \beta \in \alpha\}^B$, so that by absoluteness of $\bigcup$, we get $G_A(\alpha_0) = G_B(\alpha_0)$.

Remark: Note that we implicitly used that the Recursion Theorem holds (both existence and uniqueness). We can try a more explicit proof which will be messier:

We take $\psi_{F,a}(\alpha, g)$ and G from the Recursion Theorem. We then assert that under the assumptions

$$\forall z \in A \left(z \in G^A \leftrightarrow z \in G^B\right)$$

where

$$z \in G \equiv z \text{ is an ordered pair } \wedge \exists g\ \psi_{F,a}(\pi_1(z), g) \wedge g(\pi_1) = \pi_2(z).$$

Maybe we should get rid of the abbreviations to see that $\pi_1(z), \pi_2(z)$ are really harmless:

$$\begin{aligned} z \in G \equiv\ & \exists a, b \in z\ \exists \alpha, y \in b \\ & z = \{a, b\} \wedge a = \{\alpha\} \wedge b = \{\alpha, y\} \quad \text{(expressing that } z = \langle \alpha, y \rangle\text{)} \\ & \wedge \exists g\ \psi_{F,a}(\alpha, g) \wedge \exists w \in g\ w = z \quad \text{(the last bit expressing that } g(\alpha) = y\text{).} \end{aligned}$$

We drill down into the definition of $\psi_{F,a}$ similarly.

We then relativize **everything** to A and B respectively (even the $\alpha + 1$ and $\bigcup$) and then apply that A, B satisfy enough of ZF and are transitive to get rid of most of the relativizations (i.e. prove these subformulae are equivalent to the ones without the relativization) except for $\exists g \in A$ and $\exists g \in B$ respectively ($\alpha \in$ On is absolute by question 5 and everything else should be Δ_0, I hope).

So assume that there is $z \in G^A$. Then $z \in B$ (as $A \subseteq B$) and the $g \in A$ which witnesses $z \in G^A$ also belongs to B. But the stuff it satisfies is absolute, so it also witnesses that $z \in G^B$.

Conversely, assume that $z \in G^B \cap A$ (note that we don't actually need the $\cap A$ in this case - see below). Take the α, y and the g from G^B. Now note that $\alpha \in \mathrm{On} \subseteq A$ and $\alpha \in \mathrm{On}$ is absolute so by the proof of the recursion theorem in A there is $\hat{g} \in A$ such that $\psi_{F,a}(\alpha, \hat{g})^A$. The latter is equivalent (absoluteness) to $\psi_{F,a}(\alpha, \hat{g})^B$. By the proof of the recursion theorem in B, such a $\hat{g}$ is unique so that $g = \hat{g} \in A$ and $y = g(\alpha) = \hat{g}(\alpha) \in A$. Then $\langle \alpha, y \rangle \in G^A$ and because being the ordered pair of α and y is absolute, we have $z = \langle \alpha, y \rangle \in G^A$.

Remark: in fact we have used $\mathrm{On}^A = \mathrm{On}^B$ to show $G^A = G^B$ instead of merely $G^A = G^B \cap A$ (we never used that $z \in A$ but rather got this information out of the proof.

Question 8

1. $x \subseteq y \equiv \forall t \in x \, [t \in y]$ which is Δ_0 so absolute.

2. $z = \{x_1, \ldots, x_n\} \equiv x_1 \in z \wedge \ldots x_n \in z \wedge \forall t \in z \, [t = x_1 \vee \cdots \vee t = x_n]$ which is Δ_0 so absolute.

3. $z = \langle x_1, \ldots, x_n \rangle$: We define this by induction (in the meta-theory) as follows:

$$z = \langle\rangle \equiv z = \emptyset \equiv \forall t \in z \, [t \neq t] \tag{9}$$

$$z = \langle x_1 \rangle \equiv z = \{x_1\} \tag{10}$$

$$z = \langle x_2 \rangle \equiv z = \{\{x_1\}, \{x_1, x_2\}\} \tag{11}$$

$$z = \langle x_1, x_2, \ldots, x_{n+1} \rangle \equiv z = \langle \langle x_1, \ldots, x_n \rangle, x_{n+1} \rangle \tag{12}$$

   We could of course write out a formula for each n, but this would be painful. However, all the 'defined' notions which we use are Δ_0 so the formula we would to write down (if we were forced to do so) are Δ_0.

   The alternative is to define the two-tuple, some totally ordered set of size n (e.g. $n \in \omega$) and then $z = \langle x_1, \ldots, x_n \rangle$ by $z = \{\langle 0, x_1 \rangle, \ldots, \langle n-1, x_n \rangle\}$.

4. x is an n-tuple: The obvious definition $\exists x_1, \ldots, x_n z = \langle x_1, \ldots, x_n \rangle$ is not Δ_0. But we can be slightly tricky as follows:

$$\exists x_n, t_{n-1} \in z \exists x_{n-1}, t_{n-2} \in t_{n-1} \ldots \exists x_2, t_1 \in t_2 \exists x_1 \in t_1 \, [z = \langle x_1, \ldots, x_n \rangle] \tag{13}$$

   and this is Δ_0.

   So (the important case), z is a two-tuple would be

$$\exists x_2, t_1 \in z \exists x_1 \in t_1 \, [z = \langle x_1, x_2 \rangle] \, . \tag{14}$$

Similarly, if you define the tuple via functios, you can be crafty and write down a Δ_0 formula for a given n.

Note however that the formula 'for some n, z is an n-tuple'is tricky: the inductive definition does not work since the resulting formula would be an infinite disjunction. The definition via functions does work $\exists n \in \omega\ \ldots$ but this might not mean what you think it does: there are 'weird' structures satisfying enough of ZF in which elements of ω are not necessarily what you think they should be.

5. z is an n-tuple and $\pi_i(z) = x$: We write down the formula above but also but in $\wedge x_i = x$ and again we have absoluteness. Explicitly:

$$\exists x_n, t_{n-1} \in z \exists x_{n-1}, t_{n-2} \in t_{n-1} \ldots \exists x_2, t_1 \in t_2 \exists x_1 \in t_1 \left[z = \langle x_1, \ldots, x_n \rangle \wedge x_i = x\right] \tag{15}$$

Or we could do this inductively, saying

$$z \text{ is a 0-tuple } \equiv z = \emptyset \tag{16}$$

6. $z = x \cup y$: Either we define this as $z = \bigcup\{x, y\}$ (for $\bigcup$ see later - but this only makes sense in the presence of **Pairing**) or explicitly as

$$\forall t \in z \left[\exists w \in x \left[t \in w\right] \vee \exists w \in y \left[t \in w\right]\right] \tag{17}$$
$$\wedge \forall t \in x \left[t \in z\right] \wedge \forall t \in y \left[tinz\right] \tag{18}$$

which is Δ_0 so absolute.

7. $z = x \cap y$: We could use separation, but it is less demanding to define it as

$$\forall t \in z \left[t \in x \wedge t \in y\right] \tag{19}$$
$$\wedge \forall t \in x \left[t \in y \to tinz\right] \tag{20}$$

which is again Δ_0.

8. $z = \bigcup x$: Instead of the 'obvious'$\forall t \left[t \in z \leftrightarrow \exists y \in x \left[t \in y\right]\right]$ which is not Δ_0, we can use

$$\forall t \in z \exists y \in x \left[t \in y\right] \tag{21}$$
$$\wedge \forall y \in x \forall t \in y \left[t \in z\right] \tag{22}$$

which is Δ_0.

9. $z = x \setminus y$:

$$\forall t \in z \left[t \in x \wedge \neg \left[t \in y\right]\right] \tag{23}$$
$$\wedge \forall t \in x \left[\neg \left[t \in y\right] \to t \in z\right] \tag{24}$$

is Δ_0.

10. x is an n-ary relation on $y_1, \ldots, y_n$ (take all the y_i equal to y):

$$\forall t \in x \exists x_1 \in y_1, \ldots x_n \in y_n \, [t = \langle x_1, \ldots, x_n \rangle] \tag{25}$$

is Δ_0.

11. x is a function:

$$\forall t \in x \, [t \text{ is a 2-tuple}] \tag{26}$$
$$\wedge \forall t_1, t_2 \in x \, [\pi_1(t_1) = \pi_1(t_2) \to t_1 = t_2] \tag{27}$$

where $\pi_1(t_1) = \pi_1(t_2)$ should of course be replaced by the appropriate formula from above, namely

$$\exists w \in t_1 \exists u \in t_2 \exists x_1, x_2 \in w \exists y_1, y_2 \in u \, [t_1 = \langle x_1, x_2 \rangle \wedge t_2 = \langle y_1, y_2 \rangle \wedge x_1 = y_1] \tag{28}$$

and everything is Δ_0

12. $z = x \times y$:

$$\forall t \in z \exists x_1 \in x \exists y_1 \in y \, [t = \langle x_1, y_1 \rangle] \tag{29}$$
$$\wedge \forall x_1 \in x \forall y_1 \in \exists t \in z \, [t = \langle x_1, y_1 \rangle] \tag{30}$$

is Δ_0

13. x is a function and $dom(x) = z$:

$$x \text{ is a function} \tag{31}$$
$$\wedge \forall t \in x \ \pi_1(t) \in z \tag{32}$$
$$\wedge \forall w \in z \exists t \in x \ \pi_1(t) = w \tag{33}$$

where $\pi_1(t) \in z$ should of course be replaced by an appropriate Δ_0 formula.

14. x is a function and $ran(x) = z$: very similar to the previous one.

15. x is transitive:

$$\forall y \in x \forall t \in y \, [t \in x] \tag{34}$$

is Δ_0

16. x is an ordinal: This one is not absolute for transitive classes satisfying only ZF$^-$! See the lecture notes. However, assuming foundation, there is an equivalent definition which is absolute.

17. x is a successor ordinal:

$$x \in On \wedge \exists y \in x \, [x = y \cup \{y\}] \tag{35}$$

and this is absolute provided being and ordinal is absolute.

18. x is a limit ordinal: either x is an ordinal and not a successor ordinal or x is an ordinal and $\forall y \in x \exists z \in x\, [z = y \cup \{y\}]$. Again, this is absolute provided being an ordinal is absolute.

19. $x = \omega$:

$$x \text{ is a limit ordinal} \wedge x \neq \emptyset \tag{36}$$
$$\wedge \forall y \in x\, [y \text{ is a successor ordinal} \vee y = \emptyset] \tag{37}$$

which is absolute if being an ordinal is absolute.