Sheet 3

Question 1

1. Let $\alpha \in \mathrm{On}$. By recursion on ω, define $\alpha_0 = \alpha + 1$ and α_{n+1} as the least element of C_1 that is $> \alpha_n$ if n is even and as the least element of C_2 that is $> \alpha_n$ if n is odd. Since C_i are unbounded this is well-defined. Hence α_n is a strictly increasing sequence of ordinals. Set $\gamma = \sup_{n\in\omega} \alpha_n$. Since both α_{2n} and α_{2n+1} are unbounded in $\{\alpha_n : n \in \omega\}$ we have $\sup_{n\in\omega} \alpha_{2n} = \gamma = \sup_{n\in\omega} \alpha_{2n+1}$. But $\alpha_{2n} \in C_2$ for each $n \in \omega$ so that $\gamma \in C_2$ as C_2 is closed (under suprema). Similarly $\alpha_{2n+1} \in C_1$ for each $n \in \omega$ so that $\gamma \in C_1$. Hence $\gamma \in C_1 \cap C_2$ and by construction $\alpha < \alpha_0 < \gamma$. As $\alpha \in \mathrm{On}$ was arbitrary, $C_1 \cap C_2$ is unbounded.

   That $C_1 \cap C_2$ is closed under suprema is trivial since each of C_1, C_2 is. Formally, suppose $A \subseteq C_1 \cap C_2$ is a set of ordinals. Then by assumption $\sup A \in C_1$ as $A \subseteq C_1$ and C_1 is closed under suprema and similarly $\sup A \in C_2$ so that $\sup A \in C_1 \cap C_2$.

2. One possible formula expressing that each X_i is club is

   $$\begin{aligned} \forall i \in \omega\ \forall \alpha \in \mathrm{On}\ \exists \gamma \in \mathrm{On}\ \ [\alpha \in \gamma \wedge \langle i, \gamma\rangle \in X] \wedge \\ \forall i \in \omega\ \forall \gamma \in \mathrm{On}\ \ [[\forall \alpha \in \gamma\ \exists \beta \in \gamma\ \ \langle i, \beta\rangle \in X] \to \langle i, \gamma\rangle \in X]\,. \end{aligned}$$

   We define
   $$\bigcap_{i\in\omega} X_i = \{\alpha \in \mathrm{On} : \forall i \in \omega\ \ \langle i, \alpha\rangle \in X\}\,.$$

   Note that the obvious definition as $\bigwedge_{i\in\omega} \phi_i$ where $\phi_i(\alpha) \equiv \langle i, \alpha\rangle \in X$ is not a first-order formula (the conjunction is infinite).

   To see that this is unbounded, let $\alpha \in \mathrm{On}$. We take an explicit bijection $f : \omega \to \omega\times\omega$ (these exist), write $f(i) = \langle n_i, m_i\rangle$ and recursively define $\alpha_0 = \alpha$ and α_{i+1} to be the least element of $X_{n_{i+1}}$ that is strictly bigger than $f\alpha_i$. It is not difficult to see that for each $k \in \omega$, $\gamma = \sup\{\alpha_i : i \in \omega\} = \sup\{\alpha_i : i \in \omega, n_i = k\}$ so that $\forall k \in \omega \gamma \in X_k$.

   Alternatively, let $Y_i = \bigcap_{j\le i} X_j = \{\alpha : \forall j \le i\ \ \langle j, \alpha\rangle \in X\}$. (Really, we should set $Y = \{\langle i, \alpha\rangle : i \in \omega \wedge \forall j \le i\ \ \langle j, \alpha\rangle \in X\}$.) Note that by induction on i and the first part, each Y_i is club. Given $\alpha \in \mathrm{On}$, let $\alpha_0 \in Y_i$ be minimal such that $\alpha_0 > \alpha$ and define recursively α_{n+1} as minimal in Y_n such that $\alpha_{n+1} > \alpha_n$. Set $\gamma = \sup\{\alpha_n : n \in \omega\}$. Since the sequence of α_n are strictly increasing $\gamma = \sup\{\alpha_n : m \le n \in \omega\}$ for each $m \in \omega$ and by the definition of Y_m we then have $\gamma \in Y_m$ for each $m \in \omega$. But $Y_m \subseteq X_m$, so $\forall m \in \omega\ \ \langle m, \gamma\rangle \in X$ as required.

   Note, that in general it does not make sense to talk about the class $\bigcap_{i\in\omega} X_i$ since we cannot (in general) write down a finite formula describing it. The only reason we can do so is because we have one formula which works (by instantiation) for each X_i, so we have a uniform description of the X_i.

If we relativize everything to a transitive set M which models enough of ZF to allow recursion on ω and to show that suprema of sets (elements of M!) of ordinals exist and are ordinals (etc) and each X_i is a subclass of M (i.e. we relativize everything to a transitive set-model of enough of ZF to carry out the argument) then we can (in V) form the subclass $\bigcap_{i\in\omega} X_i$ (using $val(M,.,.)$ - it is a set in V but might not be in M) and then we have indeed proved that this is unbounded (according to M). It is then straightforward to show that it is also closed under suprema.

Question 2

Part (i): Suppose $\mathrm{ZF} \vdash \exists x\ \phi(x)$. Consider the class $\{\alpha \in \mathrm{On} : \exists x \in V_\alpha\ \phi(x)\}$. This is non-empty by assumption (as $\mathrm{ZF} \vdash \forall x \exists \alpha \in \mathrm{On}\ x \in V_\alpha$), so let α be its minimal element. Fix $x \in V_\alpha$ such that $V \models \phi(x)$. Note that by minimality of x, α is a successor ordinal $\beta + 1$ and $x \subseteq V_\beta$. Then x is transitive and $x \models \mathrm{ZF}$ so that $x \models \exists y\ \phi(y)$. Fix $y \in x$ such that $x \models \phi(y)$. As x is transitive and ϕ is absolute we have $V \models \phi(y)$. But $y \in x$ so $y \in V_\beta$, contradicting minimality of α.

Part (ii): If T was a finite subcollection of sentences such that $T \vdash \mathrm{ZF}$, then $V \models \bigwedge T$, so we can apply Levy's Reflection Principle to V_α with $\phi = \bigwedge T$ to obtain some V_α such that $V_\alpha \models \bigwedge T$. Then (by soundness and completeness) $V_\alpha \models \mathrm{ZF}$ and V_α is transitive, so that $\phi(V_\alpha)$. Hence $\exists x\ \phi(x)$ (namely V_α) and (again by soundness and completeness), $\mathrm{ZF} \vdash \exists x\ \phi(x)$. (In fact, using soundness and completeness is not necessary here. We could carry out the whole argument on the formal, syntactic side.)

Part (iii): We code the formulae of LST by natural numbers, writing $\ulcorner \phi \urcorner$ for the code of ϕ. We can also write down the set of natural numbers X which are codes for axioms. This is easy for everything except for the axiom scheme **Separation** and **Replacement**, but even for these it is fairly straightforward. Note that everything, and in particular X, will be absolute.

Finally, $\phi(x)$ would be

$$\forall n \in X\ \ \mathrm{val}(x, n, \emptyset) = 1.$$

Part (iv): It is tricky to even figure out how to formalize this question, because it does not make sense to talk about absoluteness between infinitely many V_α and V (since absoluteness between V_α and V amounts to having a proof that $\forall a_1, \ldots, a_n \in V_\alpha\ \left[\phi^{V_\alpha} \leftrightarrow \phi^V\right]$ and we only ever have finitely many proofs).

Thus, what is asserted here might be: For every formula $\phi(v_1, \ldots, v_n)$ of LST,

$$\mathbf{ZF} \vdash \forall \alpha \in \mathrm{On}\ \exists \gamma \in \mathrm{On}\ \left[\forall a_1, \ldots, a_n \in V_\gamma \left[\phi(a_1, \ldots, a_n)^{V_\gamma} \leftrightarrow \phi(a_1, \ldots, a_n)^V\right]\right]$$

and

$$\mathbf{ZF} \vdash \forall\gamma \in \mathrm{On}$$
$$\Big[\forall\alpha \in \gamma \exists\beta \in \gamma\ \big[\alpha \in \beta \wedge \forall a_1, \ldots, a_n \in V_\beta\ \big[\phi(a_1, \ldots, a_n)^{V_\beta} \leftrightarrow \phi(a_1, \ldots, a_n)^{V_\gamma}\big]\big]\Big]$$
$$\rightarrow$$
$$\forall a_1, \ldots, a_n \in V_\gamma\ \big[\phi(a_1, \ldots, a_n)^{V_\gamma} \leftrightarrow \phi(a_1, \ldots, a_n)^{V}\big].$$

The first of these is the Levy Reflection Principle.

For the second, we define (in the meta-theory) by recursion on the complexity of formulae

$$C_\phi = \mathrm{On}$$

for ϕ atomic,

$$C_{\neg\phi} = C_\phi$$
$$C_{\phi\wedge\psi} = C_\phi \cap C_\psi$$

and

$$C_{\exists x\ \phi} = C_\phi \cap \big\{\alpha \in \mathrm{On} : \forall a_1, \ldots, a_n \in V_\alpha\ \big[\exists x \in V\ \phi(a_1, \ldots, a_n, x)^V \rightarrow \exists x \in V_\alpha\ \phi(a_1, \ldots, a_n)^V\big]\big\}.$$

By induction on the complexity of the formula we show that each C_ϕ is club. For the existentially quantified case, apply the Tarski-Vaught criterion.

Remark: I think the following formula shows that in general the class

$$\{\alpha \in \mathrm{On} : \phi \text{ is absolute for } V_\alpha, V\}$$

is not club.

We let ϕ be the sentence expressing that there is a set containing every finite ordinal, i.e.

$$\phi \equiv \exists z\ \forall\beta \in \mathrm{On}\,[\beta \text{ is finite } \rightarrow \beta \subseteq z].$$

Clearly ϕ^V is true as witnessed by ω. For every $n \in \omega$, we have $\mathrm{On}^{V_{n+1}} = \mathrm{On} \cap V_{n+1} = n + 1$ and $z = n \in V_{n+1}$ witnesses ϕ. But no $z \in V_\omega$ witnesses ϕ (it would be ω, but $\omega \notin V_\omega$).

Part (v): There are various problems:

- The first problem is that $\bigcap_{i\in\omega} C_i$ is not defined (see above). Again, we can fix this if we go to the meta-theory: we define $C = \{\langle n, \alpha\rangle : n \in Form \wedge val(n, \emptyset, V_\alpha) = 1\}$ which contains each C_i.
- The real problem is the LRP: to apply it to all the ϕ_i at once, we would need to be able to define $val(.,., V)$.

  For each i, the LRP spits out a proof P_i from ZF that $C_{\phi_i} = \{\langle i, \alpha\rangle : val(i, \emptyset, V_\alpha) = 1\}$ is club. But since for each i a different proof is given, it does not give a finite proof that $\forall i \in Form\ C_i$ is club.

- You could of course try to 'internalize' the notion of 'there is a proof' roughly as follows: coding formulae by natural numbers, saying that there is a proof of ϕ amounts to saying that there is a sequence of codes with specific properties (as dictated by the logical calculus you are using) ending in $\lceil\phi\rceil$. Then you could write down a statement in LST which seems to mean 'for each ϕ there is a proof of C_{ϕ_i} is a club'. But in doing so, you have to be careful: in non-standard models of ZFC, there could be non-standard natural numbers and hence your internalization does not mean what you intend it to mean.

Question 3

I've done the proofs of $L \models ZF$ in lectures except for **Union** and **Infinity**.

Extensionality L is transitive, so $L \models$ **Extensionality**.

Separation This has been done in lectures - use the Reflection Theorem.

Note that if we have the Reflection Theorem for the hierarchy $L_\beta, \beta \in \mathrm{On}^{L_\alpha} = \mathrm{On} \cap L_\alpha = \alpha$, then **Separation** will hold in L_α. The proof of the Reflection Theorem goes through, provided that α is a limit ordinal such that every countable supremum of ordinals $< \alpha$ is $< \alpha$, i.e. that $\mathrm{cf}\alpha > \omega$.

Pairing If $x, y \in L$ then find $\alpha \in \mathrm{On}$ such that $x, y \in L_\alpha$. Then $z = \{t \in L_\alpha : L_\alpha \models t = x \vee t = y\} \in Def(L_\alpha) = L_{\alpha+1}$ and absoluteness shows that $L \models z = \{x, y\}$ (i.e. $L \models x \in z \wedge y \in z \wedge \forall t \in z\ [t = x \vee t = y]$).

We note that this in fact shows that if $\alpha \in \mathrm{Lim}$ then $L_\alpha \models$ **Pairing**.

Union If $x \in L$ then find $\alpha \in \mathrm{On}$ such that $x \in L_\alpha$. Then

$$z = \{t \in L_\alpha : L_\alpha \models \exists y \in x\ t \in y\} \in Def(L_\alpha) = L_{\alpha+1}$$

and by absoluteness $L \models z = \bigcup x$ (i.e.

$$L \models [\forall t \in z \exists y \in x\ tx \in y] \wedge [\forall y \in x \forall t \in y\ t \in z]$$

).

Again, this shows that for $\alpha \in \mathrm{Lim}$, $L_\alpha \models$ **Union**.

Replacement Assume that $\phi(x, y; v_0, \ldots, v_n)$ is a formula of LST and $a_0, \ldots, a_n, d \in L$ such that $L \models \forall x \in d \exists! y\ \phi(x, y, a_0, \ldots, a_n)$ i.e. we assume $\forall x \in d \cap L \exists! y \in L\ \phi(x, y, a_0, \ldots, a_n)^L$.

Let $\psi(x, y; v_0, \ldots, v_n) \equiv y \in L \wedge \phi(x, y, a_0, \ldots, a_n)^L$. Since L is transitive and $d \in L$, $d \cap L = d$. Then $V \models \forall x \in d \exists! y\ \psi(x, y, a_0, \ldots, a_n)$ (by substituting ψ), so apply Replacement in V so that we find $z \in V$ such that

$$z = \{y : \exists x \in d\ \psi(x, y, a_0, \ldots, a_n)\}$$

and note (substitute and push the relativization out using $d \cap L = d$)

$$z = \left\{ y \in L : [\exists x \in d\ \phi(x, y, a_0, \ldots, a_n)]^L \right\}.$$

Observe that $z \subseteq L$ and hence note that by **Replacement** and **Union** in V, for $y \in z$ we can find $\alpha_y \in \mathrm{On}$ minimal such that $y \in L_{\alpha_y}$ and by setting $\alpha = \sup \{\alpha_y : y \in z\}$ $z \subseteq L_\alpha$.

Now we can apply **Separation** in L to see that

$$z' = \left\{ y \in L_\alpha : [\exists x \in d\phi(x, y, a_0, \ldots, a_n)]^L \right\} \in L$$

and check that z' is as required, i.e. that $[\forall t\, [t \in z' \leftrightarrow \exists x \in d\phi(x, y, a_0, \ldots, a_n)]]^L$: $\rightarrow$ is clear from the definition of z'. For $\leftarrow$ we may note that if $t \in L$ such that $\exists x \in d\phi(x, y, a_0, \ldots, a_n)^L$ then $t \in z \subseteq L_\alpha$, so $t \in z'$.

Alternatively (to avoid **Separation**) we check manually (using Reflection) that $z \in L$: for this first increase α so that $a_0, \ldots, a_n, d \in L_\alpha$ as well as $z \subseteq L_\alpha$. Then apply the Reflection Theorem to find $\gamma > \alpha$ such that $\exists x \in d\ \phi(x, y, v_0, \ldots, v_n)$ is absolute for L_γ, L. Hence

$$z = \left\{ y \in L_\alpha : [\exists x \in d\ \phi(x, y, a_0, \ldots, a_n)]^{L_\alpha} \right\} \in Def(L_\alpha).$$

Now by construction, z is as required, i.e. $[\forall t\, [t \in z \leftrightarrow \exists x \in d\ \phi(x, y, a_0, \ldots, a_n)]]^L$.

To have **Replacement** true in L_α, we want **Separation** (or the Reflection Theorem 'up to L_α'). We also somehow need to be able to prove that the z we construct above is a subset of L_α. To do so we want a result along the lines: if $d \in L_\alpha$ and $f : d \rightarrow \alpha = \mathrm{On}^{L_\alpha}$ is a function then $\sup f < \alpha$.

Powerset Suppose $x \in L$ and use **Powerset** to find $z \in V$ such that $V \models z = \mathcal{P}(x)$, i.e. such that $\forall t\, [t \in z \leftrightarrow t \subseteq x]$. Let $z' = z \cap L$ (by **Separation** in V). Then $z' \subseteq L$ and as in the proof for Replacement, we can find $\alpha \in \mathrm{On}$ such that $z' \subseteq L_\alpha$ and $x \in L_\alpha$. Hence

$$z' = \left\{ y \in L_\alpha : [y \subseteq x]^L \right\} = \left\{ y \in L_\alpha : [y \subseteq x]^{L_\alpha} \right\} \in Def(L_\alpha) = L_{\alpha+1} \subseteq L,$$

using absoluteness of $\subseteq$. Again, using absoluteness of $\subseteq$ it is now easy to check that $L \models z' = \mathcal{P}(x)$, i.e. $[\forall t\, [t \in z' \leftrightarrow t \subseteq x]]^L$ holds.

For **Powerset**, even if $x \in L_\alpha$, $\mathcal{P}(x)^L$ can have arbitrarily large rank, so (short of inaccessible cardinals), I can't find specific $\alpha > \omega$ for which L_α satisfies **Powerset** (although by the Reflection Theorem there must be lots of them and in principle it should be possible to write one down explicitly).

Infinity Either note that $\omega \in L_{\omega+1}$ (a formula defining ω in L_ω is $\phi(t) \equiv t \in \mathrm{On}$) or that in fact $L_\omega \in L_{\omega+1}$ and L believes that both ω and L_ω are inductive and non-empty.

Clearly for every $\alpha > \omega$ we have that $L_\alpha \models$ **Infinity**.

Foundation L is a transitive subclass of V.

Part 3:

Of course, I missed '$\omega^V \in A$' in the question statement. As stated, A might not satisfy **Infinity**, e.g. if $A = V_\omega$.

Transitivity shows **Extensionality** and **Foundation** is downwards absolute. For **Pairing** and **Union**, note that if $x, y \in A$ then $z = \{x,y\}^V \subseteq A$ and $z = (\bigcup x)^V \subseteq A$. By assumption $z \in A$ and by absoluteness of $\{x,y\}$ and $\bigcup x$, $z = \{x,y\}^A$ and $z = (\bigcup x)^A$. For **Powerset** note that if $x \in A$ then $z = \mathcal{P}(x)^V \cap A \subseteq A$ so $z \in A$ and $z = \mathcal{P}(x)^A$.

This leaves **Replacement** where you set $z = \left\{y : \exists x \in d\ y \in A \wedge \phi(a_1, \ldots, a_n, x, y)^A\right\}^V$, note $z \subseteq A$ so $z \in A$ and check $z = \{y : \exists x \in d\phi(a_1, \ldots, a_n, x, y)\}^A$.

Question 4

We have shown in lectures that $\alpha \subseteq L_\alpha \subseteq V_\alpha$ and $V_\alpha \cap \mathrm{On} = \alpha$ on a problem sheet. $L_\alpha \subseteq V_\alpha$ shows that $rk(L_\alpha) \geq \alpha$. But for $\beta < \alpha$, $\alpha \not\subseteq \beta = V_\beta \cap \mathrm{On}$ so $rk(L_\alpha) > \beta$. Thus the result follows.

Question 5

There are at least two ways to achieve this: the first is that the recursion theorem (used to define ordinal addition) gives an explicit formula $\phi(z)$ such that $\phi(z)$ if and only if z is a pair $\langle \alpha, \beta \rangle$ of ordinals and $\beta = \alpha + \alpha$. We then set $\psi(x) \equiv x \in \mathrm{On} \wedge \exists y \in \mathrm{On}\ \phi(\langle y, x \rangle)$. Noting that $\mathrm{On}^{L_\omega} = \mathrm{On} \cap L_\omega = \omega$ we do have

$$E = \left\{t : t \in L_\omega \wedge \psi(t)^{L_\omega}\right\}.$$

If you are worried by $\mathrm{On}^{L_\omega} = \omega$, then you can of course also include the absolute formula $x \in \omega$, which is shorthand for $x = \emptyset$ or (x is a successor ordinal and all elements of x are successor ordinals). You cannot leave ω as a parameter, since $\omega \notin L_\omega$.

Alternatively, you write down an absolute formula $\phi(x)$ that expresses: each element of x is a set of two distinct elements and any two elements of x are pairwise disjoint. For example

$$\forall t \in x \exists a, b \in t\ t = \{a, b\} \wedge a \neq b$$
$$\forall t, t' \in x\ t \cap t' = \emptyset$$

replacing the shorthand $t = \{a, b\}$ and $t \cap t' = \emptyset$ by Δ_0 formulae respectively.

Then $\psi(t) \equiv t \in \mathrm{On} \wedge \exists x\ \psi(x) \wedge t = \bigcup x$ is the required formula (note the absoluteness of this), because as before $\mathrm{On}^{L_\omega} = \omega$.

Question 6

The quick solution is to show:

F is injective: Consider $\phi(x, y) \equiv x = y$. Then $F(x) = F(y) \leftrightarrow \phi(F(x), F(y)) \leftrightarrow \phi(x, y) \leftrightarrow x = y$.

F is surjective: Consider $\phi(y) \equiv \exists x\ y = F(x)$. If $y \in V$ then $F(y) = F(y)$ so $\phi(F(y))$ holds, hence $\phi(y)$ holds.

F is the identity: Observe that $\phi(y, x) \equiv y \in x$ gives $y \in x \leftrightarrow F(y) \in F(x)$. Thus $F(x) = \{t : t \in F(x)\} = \left\{F^{-1}(u) : F(u) \in F(x)\right\} = \left\{F^{-1}(u) : u \in x\right\}$. Now by $\in$-induction, considering the least x such that $F(x) \neq x$, gives $F(x) = x$ since by minimality $u \in x \to F^{-1}(u) = u$.

Alternative Solution: $\phi(x) \equiv x \in \mathrm{On}$ is preserved so $F[\mathrm{On}] = \mathrm{On}$.

Now consider $\phi(x) \equiv x \neq F(x) \wedge x \in \mathrm{On} \wedge \forall y \in x\ F(y) = y$ expressing that x is the least ordinal that is not preserved by F. Assume now that there is some ordinal α such that $F(\alpha) \neq \alpha$. Choosing α minimal and write $\beta = F(\alpha)$. Then $\phi(\alpha)$ (by minimality of α), so that y holds. Since $\alpha, \beta \in \mathrm{On}$ and $\beta = F(\alpha) \neq \alpha$ we must have one of $\alpha \in \beta$ or $\beta \in \alpha$. If $\alpha \in \beta$ then the last conjunct in $\phi(\beta)$ fails. If $\beta \in \alpha$ then since by the first conjunct in $\phi(\beta)$ we have $\beta \neq F(\beta)$, the last conjunct in $\phi(\alpha)$ fails.

Now consider $\psi(x, \alpha) \equiv \alpha \in \mathrm{On} \wedge rk(x) = \alpha$. Since $\alpha = F(\alpha)$ for ordinals α we obtain that $rk(x) = rk(F(x))$ for all x.

Finally, assume that F is not the identity and choose x such that $x \neq F(x)$ and the rank of x is minimal. Then for each $y \in x$ we have $F(y) = y$ and considering $\theta(x, y) = y \in x$ we obtain (as in the first part) that $F(x) = \{F(y) : y \in x\} = \{y : y \in x\} = x$.

Note: It is critical that F is given by an explicit formula so that we obtain some sort of self-referential formula (surjectivity in the first solution and ϕ in the second solution). Assuming (for the sake of argument) that some V_κ (or L_κ) is a model for **ZF** it is perfectly conceivable that there is $f : V_\kappa \to V_\kappa$ such that f is a non-trivial elementary embedding of V_κ into itself. It is just that we cannot find an explicit formula for f, so the above proof fails.

Question 7

We show something stronger, namely that if $\phi(x_0, \ldots, x_n)$ is Σ_1 (i.e. all universal quantifiers are bounded) then there is a Δ_0 formula $\psi(x_0, \ldots, x_n, y)$ such that

$$\mathbf{ZF} \vdash \forall a_0, \ldots, a_n \left[\phi(a_0, \ldots, a_n) \leftrightarrow \exists y \psi(a_0, \ldots, a_n, y)\right].$$

The proof is by induction on the complexity of ϕ. First wlog all quantifiers occur at the front of ϕ (make all dummy variables different and push them to the front).

Base Case: If ϕ is Δ_0 then take $\psi = \phi$ and we are done.

Inductive Step: Conjunction and disjunction is trivial. So assume that $\phi \equiv \exists x_{n+1}\phi'(x_0,\dots,x_{n+1})$. By inductive hypothesis, there is a Δ_0 formula $\psi'(x_0,\dots,x_{n+1},y')$ such that $\phi'(x_0,\dots,x_{n+1})$ is equivalent (in **ZF**) to $\exists y'\psi'$. We may thus assume wlog that $\phi \equiv \exists x_{n+1}\exists y'\psi'(x_0,\dots,x_{n+1},y')$. We let $\psi(x_0,\dots,x_n,y)$ be

$$\exists x_{n+1} \in y \exists y' \in y \;\; [y = \{x_{n+1}, y'\} \wedge \psi'(x_0,\dots,x_{n+1},y')]$$

and observe that this clearly works (since $y = \{x_{n+1}, y'\}$ is really the Δ_0-formula $x_{n+1} \in y \wedge y' \in y \wedge \forall t \in y\, [t = x_{n+1} \vee t = y']$).

Next assume (using the inductive hypothesis and an argument as above) that $\phi \equiv \forall x_{n+1} \in x_n \exists y'\psi'(x_0,\dots,x_{n+1},y')$ for some Δ_0 formula ψ. Let

$$\psi(x_0,\dots,x_n,v) \equiv \forall x_{n+1} \in x_n \exists y' \in v \;\psi'(x_0,\dots,x_{n+1},y').$$

We need to verify that

$$\mathbf{ZF} \vdash \forall a_0,\dots,a_n\, [\phi(a_0,\dots,a_n) \leftrightarrow \exists v \;\psi(a_0,\dots,a_n,v)]$$

which, written out says

$$\begin{aligned}\mathbf{ZF} \vdash &\forall a_0,\dots,a_n \\ &\qquad \forall x_{n+1} \in a_n \exists y' \;\psi'(x_0,\dots,x_{n+1},y') \\ &\qquad\quad \leftrightarrow \\ &\qquad\qquad \exists v \forall x_{n+1} \in a_n \exists y' \in v \;\psi'(x_0,\dots,x_{n+1},y')\end{aligned}$$

So fix $a_0,\dots,a_n$. First assume that $\phi(a_0,\dots,a_n)$ holds. For each $x \in a_n$ find $\alpha_x \in \mathrm{On}$ minimal such that $\exists y' \in V_{\alpha_x}\psi'(x_0,\dots,x,y')$ (there is such a $y' \in V$, so there is such a y' in some V_α). Let $\alpha = \sup\{\alpha_x : x \in a_n\}$ and note that by construction and $V_{\alpha_x} \subseteq V_\alpha = v$ for $x \in a_n$ so do have $\forall x_{n+1} \in a_n \exists y' \in V_\alpha \;\psi'(a_0,\dots,a_n,y')$ as required.

The converse is obvious (on the LHS there is no restriction on the y', whereas on the RHS there is).