Axiomatic Set Theory Sheet 1

1. Interpret the strict total order $(\mathbb{Q}, <)$ a a model of the language of set theory (i.e. we interpret the binary predicate $\in$ as $<$).

Which of the axioms **Extensionality**, **Emptyset**, **Pairing**, **Union**, **Powerset**, **Infinity** hold? Find an instance of **Separation** that is true and one that is false.

Do the same for the substructure $(\mathbb{N}, <)$.

Optional: For an arbitrary strict partial order $(P, <)$, find the order theoretic interpretations of $\emptyset$, $\{x, y\}$, $\bigcup x$, $\mathcal{P}(x)$ and conditions of their existence. Given a substructure $(Q, <)$ of $(P, <)$ under what conditions on Q (and P) are these absolute?

2. Assuming $\mathbf{ZF}^-$, show that there exists a transitive set M such that

1. $\emptyset \in M$;
2. $\forall x, y \in M \ \{x, y\} \in M$;
3. every element of M contains at most two elements.

Show that $(M, \in)$ models **Extensionality**, **Emptyset**, **Pairing**, **Separation**, **Replacement** but not **Union**, **Powerset**, **Infinity**.

Does $(M, \in)$ model (your favourite version of) **Choice**?

3. Assuming **ZF** show that is a is a non-empty transitive set then $\emptyset \in a$.

Explain why the following sketch proof is not correct: Suppose $\emptyset \notin a$. By recursion on $n \in \omega$ find $x_n \in a$ such that $x_{n+1} \in x_n$ (in the inductive step we are using that $x_n \in a$ means that $x_n \neq \emptyset$ and then transitivity of a to get $x_{n+1} \in a$ as well). Then $\{x_n : n \in \omega\}$ is a subset of a with no minimal element, contradicting **Foundation**.

4. Deduce Pairing from the other axioms of $\mathbf{ZF}^-$.

5. Assuming **ZF** prove that the following two definitions of "ordinal" are equivalent:

1. An ordinal is a transitive set well-ordered by $\in$.
2. An ordinal is a transitive set totally ordered by $\in$.

Hence show that 'x is an ordinal' is absolute for non-empty transitive classes $A \subseteq B$ satisfying (enough of) **ZF**.

6. Formally phrase the following claim and indicate its proof:

If $A \subseteq B$ are non-empty transitive classes satisfying (enough of) **ZF**, F is a class function that is absolute for A, B and $a \in A$ then class function given by Recursion on On is absolute for A, B.

7. Read the 'Satisfaction' document to recall how we could define $(x, \in) \models \phi(x_1, \ldots, x_n, t)$ (in ZF) for a formula ϕ of LST.

8. sort of optional This question seems long but should not be difficult (most should be one line, some have been done in the lecture notes). It is enough if you are convinced you could write down appropriate formulae. Please only hand those in where you are unsure and let your class tutor know about any which you find tricky.

Give formulae of LST which express (are **ZF**-equivalent to) the following:

1. $x \subseteq y$
2. $z = \{x_1, \ldots, x_n\}$
3. $z = \langle x_1, \ldots, x_n \rangle$
4. x is an n-tuple
5. z is an n-tuple and $\pi_i(z) = x$
6. $z = x \cup y$
7. $z = x \cap y$
8. $z = \bigcup x$
9. $z = x \setminus y$
10. x is an n-ary relation on y
11. x is a function
12. $z = x \times y$
13. x is a function and $dom(x) = z$
14. x is a function and $z = ran(x)$
15. x is transitive
16. x is an ordinal
17. x is a successor ordinal
18. x is a limit ordinal
19. $x = \omega$

Which of these are (obviously) absolute for transitive non-empty classes $A \subseteq B$ satisfying (enough of) **ZF**?