Axiomatic Set Theory: Problem sheet 2

1. Ensure that you can show the facts about ordinals that we use (section 6 in the Lecture Notes).

2. (ZF^-) Define a "natural" ordinal exponentiation using the recursion theorem for ordinals, and show that for all ordinals α, β and γ, $\alpha^{(\beta+\gamma)} = \alpha^\beta \alpha^\gamma$, and $\alpha^{(\beta\cdot\gamma)} = (\alpha^\beta)^\gamma$. Show also that $2^\omega = \omega$.

3. (ZF^-) Suppose $F : On \to On$ is a class term satisfying:
(1) $\alpha < \beta \to F(\alpha) < F(\beta)$ (for $\alpha, \beta \in On$)
(2) $F(\delta) = \bigcup_{\alpha<\delta} F(\alpha)$ (for limit ordinals δ).
Prove that for all $\alpha \in On$ there exists $\beta \in On$ such that $\beta > \alpha$ and $F(\beta) = \beta$ (ie. F has arbitrarily large fixed points). What is the smallest non-zero fixed point of the term $F : On \to On$ defined by $F(x) = \omega.x$ (for $x \in On$)?

4. (ZF) Let H_ω denote the class of *hereditarily finite sets*, ie. $H_\omega = \{x : TC(x) \text{ is finite}\}$. Prove that $H_\omega = V_\omega$ (and hence that H_ω is a set). Prove in ZF^- that $\langle V_\omega, \in\rangle \vDash$ the axiom of foundation, and $\langle V_\omega, \in\rangle \vDash \neg$ the axiom of infinity.

[It is easy, but tedious, to check that $\langle V_\omega, \in\rangle \vDash$ the other axioms of ZF. This shows that the other axioms of ZF do not imply the axiom of infinity.]

5. (ZF^-) Prove that the axiom of foundation is equivalent to $\forall x(x \in V)$.

6. Complete the proof that $(V, \in) \vDash$ ZF (i.e. prove the axioms which were skipped in the lectures - this will probably be Union and Infinity).

7. Prove that $\forall \alpha, \beta \in On$, (i) $V_\alpha \cap On = \alpha$, and (ii) if $\alpha \in V_\beta$, then $V_\alpha \in V_\beta$.

8. Strictly Optional: I will at least outline the proof below in the lectures, but it is instructive to prove this version of the Recursion Theorem yourself.

Work in $\mathbf{ZF}^-$.

For a relation R on a class A, we say that R is set-like if

$$\forall a \in A \ \exists z \ \forall b \in A \ (bRa \to b \in z),$$

i.e. if $pred(a) = \{b : bRa\}$ is a set.

As always, we write $U = \{x : x = x\}$.

Prove the generalized recursion theorem: Suppose R is a well-founded, set-like relation on a class A and that B is a class.

If $F : A \times U \to B$ is a class function then there is a unique class function $G : A \to B$ such that for all $a \in A$,

$$G(a) = F(a, G|_{pred(a)})$$

and write down an explicit formula defining G.

Deduce the usual Recursion Theorem (on ω) from the Generalized Recursion Theorem (i.e. give an explicit R and F that 'works').

Observe which instances of Replacement are needed.

Hint/Outline: You may pretend that $A = B = U$ (why?).

First define the transitive closure of R as $R^\star$ (i.e. $R^\star$ is the 'smallest' relation containing R which is transitive) and show (or assume if this is difficult) that if R is set-like and well-founded then so is $R^\star$. We also write $pred_\star(x)$ for the $R^\star$-predecessors of x.

Next, your formula $\psi(x, g)$ should be something along the lines of 'g is a function on $pred_\star(x) \cup \{x\}$ which satisfies $g(a) = F(a, g|_{pred(x)})$ for all $a \in A \cap dom(g)$'. Now show that for every $x \in A$ there is a unique function g with $\psi(x, g)$ by 'induction ' on R.

Finally write down the formula defining G and check that it works.