Question 1

A *club* (in On) is a closed, unbounded class of ordinals, i.e. a class C such that $\forall x\ [x \subseteq C \to \sup x \in C]$ and $\forall \alpha \in \mathrm{On}\ \exists \beta \in C\ \alpha \subseteq \beta$.

1. Prove that if C_1 and C_2 are clubs then so is $C_1 \cap C_2$.

2. Suppose that X is a class such that $X \subseteq \omega \times \mathrm{On}$. For $i \in \omega$, let $X_i = \{\alpha \in On : \langle i, \alpha \rangle \in X\}$.

   Carefully write down **one** formula expressing that for all $i \in \omega$, X_i is a club.

   Carefully define $\bigcap_{i\in\omega} X_i$ and prove that it is a club.

Question 2

It is known that there is a formula $\phi(x)$ of LST (with all free variables shown) such that (in ZF one can prove that) for any set a, "$\phi(a)$ iff $(a, \in) \models \mathrm{ZF}$ and a is transitive". Further, this formula is A-absolute for any non-empty transitive class A.

1. Show that if ZF is consistent, then one cannot prove the sentence $\exists x\ \phi(x)$ from ZF.

   [Hint: Consider the least $\alpha \in On$ such that $\exists x \in V_\alpha(\phi(x))$.]

2. As formulated in the lectures, ZF is a countably infinite collection of axioms (since there is one separation and replacement axiom for each formula of LST, and there are clearly a countably infinite number of such formulas). Assuming that ZF is consistent, prove that there is no finite subcollection, T, say, of axioms of ZF, such that $T \vdash \mathrm{ZF}$.

3. (Optional) Indicate how to write down ϕ.

4. If ϕ is a formula of LST, show that the class $\{\alpha \in \mathrm{On} : \phi \text{ is absolute for } V_\alpha, V\}$ contains a club C_ϕ.

5. (Strictly Optional) What is wrong with the following argument:

   Let $\phi_i, i \in \omega$ be an enumeration of all the axioms of ZF (in a sufficiently strong meta-theory). By the previous part, for each $i \in \omega$, the class $\{\alpha \in On : (V_\alpha, \in) \models \phi_i\}$ contains a club $C_i = C_{\phi_i}$ (since $(V, \in) \models \phi_i$, i.e. $\mathrm{ZF} \vdash \phi_i^V$). By question 1, $\bigcap_{i\in\omega} C_i$ is a club. In particular, $\bigcap_{i\in\omega} C_i$ is non-empty. Let $\beta \in \bigcap_{i\in\omega} C_i$. Then $\beta \in C_i$ for all $i \in \omega$, so $(V_\beta, \in) \models \phi_i$ for all $i \in \omega$, so $(V_\beta, \in) \models \mathrm{ZF}$. Hence $\phi(V_\beta)$ holds, so $\exists x \phi(x)$ (where $\phi(x)$ is the formula from the first part). Thus ZF (by the first part) is inconsistent.

Question 3

1. Complete the proof that L satisfies ZF (again, **Union** and **Infinity**).
2. (Optional) Which axioms does L_α (obviously) satisfy (for various $\alpha \in On$). What facts would you need to show that L_α (for some appropriate α) satisfies **Replacement** and **Separation**?
3. Indicate how to show that if A is a transitive non-empty class satisfying **Separation** such that $\forall z[z \subseteq A \to z \in A]$ then A satsifies ZF.

Question 4

The *V-rank* of a set A, $\mathrm{rk}_V(A)$, is defined to be the least $\alpha \in On$ such that $A \in V_{\alpha+1}$. Prove that $\forall \alpha \in On(rk(L_\alpha) = \alpha)$.

Question 5

Let E denote the set of even natural numbers. Prove that $E \in L_{\omega+1}$.

Question 6

Suppose $F : V \to V$ is a class function without parameters (ie. the formula defining "$F(x) = y$" has no parameters). Suppose further that it is an *elementary map*, ie. for any formula $\phi(v_0, \ldots, v_n)$ of LST (without parameters), and any $a_0, \ldots, a_n \in V$,

$$\phi(a_0, \ldots, a_n) \Leftrightarrow \phi(F(a_0), \ldots, F(a_n)).$$

Prove that F is the identity. [Hint: first show that for all ordinals α, $F(\alpha) = \alpha$, by considering the first β for which $F(\beta) \neq \beta$.]

[Remark: Assuming only ZF, it is not known whether such an elementary map definable *with* parameters can exist other than the identity, although if ZFC is assumed it is known that there is no such.]

Question 7

The collection of Σ_1 formulae are defined (recursively, in the meta-theory) as follows:

- Δ_0 formulae are Σ_1;
- if ϕ and ψ are Σ_1 then so are $\phi \vee \psi$, $\phi \wedge \psi$, $\forall x \in y\ \phi$ and $\exists x\ \phi$;
- nothing else is a Σ_1 formula.

Show that for every Σ_1 formula $\phi(x_1, \ldots, x_n)$, there exists a corresponding Δ_0 formula $\psi(x_1, \ldots, x_n, y_1, \ldots, y_m)$ such that

$$ZF \vdash \forall x_1, \ldots x_n \ \left[\phi(x_1, \ldots, x_n) \leftrightarrow \exists y_1, \ldots, y_m \ \psi(x_1, \ldots, x_n, y_1, \ldots, y_m)\right].$$

Hence show that if $A \subseteq B$ are non-empty transitive classes and $\phi(v_1, \ldots, v_n)$ is a Σ_1 formula then ϕ is 'upwards absolute' for A, B, i.e.

$$ZF \vdash \forall x_1, \ldots, x_n \in A[\phi(a_1, \ldots, a_n)^A \rightarrow \phi(a_1, \ldots, a_n)^B]$$

Give an example of a Σ_1 formula that is not absolute (for non-empty transitive classes).