Axiomatic Set Theory: Problem sheet 4

Update 02.03.2017: minor corrections (replaced ω in Q1 by **On** and clarified the assumptions in Q2).

1. Assume ZF. For a set A, define $L[A]$ by recursion on **On** by $L_0 = TC(\{A\})$, $L[A]_{\alpha+1} = Def(L[A]_\alpha)$ and $L[A]_\gamma = \bigcup_{\beta<\gamma} L[A]_\beta$. Finally set $L[A] = \bigcup_{\alpha\in On} L[A]_\alpha$ (so really $L[A]$ is a formula with one free variable x saying $\exists\alpha \in On\ x \in L[A]_\alpha$).

Show that $L[A] \models ZF$, and that if A is a set of ordinals then $L[A] \models ZFC$.

Show that if $A \subseteq \omega$ then $L[A] \models CH$.

* Show that if $A \subseteq \omega_1$ and $V = L[A]$ then $L[A] \models CH$ by showing that $\mathcal{P}(\omega) \subseteq \bigcup\{L[A\cap\beta]_\alpha : \alpha, \beta \in \omega_1\}$

2. Assume **ZF** + "$\mathbf{V} = \mathbf{L}$". Show that for ordinals $\alpha > \omega$, $L_\alpha = V_\alpha$ if and only if $\alpha = \aleph_\alpha$. Show that there are ordinals α with $\alpha = \aleph_\alpha$.

3. Prove that for any infinite cardinal κ, $cf(\kappa)$ is a regular cardinal. Show that every successor cardinal κ^+ is regular.

4. Suppose κ, λ are infinite cardinals such that $\kappa \geq \lambda$. Prove that if $\lambda \geq cf(\kappa)$, then $\kappa^\lambda > \kappa$. Suppose now that $\lambda < cf(\kappa)$, and that κ has the property that for any cardinal μ, if $\mu < \kappa$ then $2^\mu \leq \kappa$. Prove that $\kappa^\lambda = \kappa$. Hence show that if GCH is assumed, then for any infinite cardinals κ, λ with $\kappa \geq \lambda$, we have $\kappa^\lambda = \kappa$ or κ^+.

5. Suppose κ is an *uncountable regular* cardinal. Let $g : \kappa \to \kappa$ be any function. Prove that for any $\alpha < \kappa$, there exists $\beta < \kappa$, with $\alpha \leq \beta$, such that β is closed under g (i.e. for all $\gamma < \beta$, $g(\gamma) < \beta$).

6. (Optional) Let κ be an uncountable regular cardinal with the property that for any cardinal $\mu < \kappa$, we have $2^\mu < \kappa$. (*).

Prove that (i) if α is any cardinal and $\alpha < \kappa$, then $|V_\alpha| < \kappa$, (ii) $|V_\kappa| = \kappa$, (iii) $\langle V_\kappa, \in\rangle \models$ ZFC.

(For (iii) you need consider only the replacement scheme, since we essentially showed that if α is a limit ordinal and $\alpha > \omega$, then $\langle V_\alpha, \in\rangle$ satisfies all the axioms of ZFC except, possibly, replacement.)

Deduce that in ZFC one cannot prove the existence of a cardinal that satisfies (*).